# TOWARDS THE THEORY OF PROOF-RELEVANT CATEGORIES

by

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## Abstract

We introduce a general framework of *proof-relevant algebras* for $T$-categories, where $T$ is the free strict $\omega$-category monad on globular sets. Through some general considerations about enrichment and internalisation in virtual double categories we give a shape functor $\mathbb{S}\colon \omega\mathrm{Cat} \to T\text{-}\mathrm{Cat}$ and prove a rich comparison of algebras under this functor. Using this framework and some supporting theory we explore definitions of *proof-relevant category* and *proof-relevant functor*, which are proof-relevant algebras of the shape of the strict $\omega$-categories 1 and $\{\cdot \to \cdot\}$ respectively. Proof-relevant categories are higher-category-like structures in which composition is multi-valued, unbiased, witnessed by proofs, and there is a calculus of proofs. The definition we give for proof-relevant category is a variation of Leinster's model $L'$ [Lei01]. Proof-relevant functors are the natural adaption of functor to the proof-relevant setting: mapping is multi-valued and unbiased, and values are witnessed by proofs. In exploring these definitions we give some of the early theory of co-inductive equivalences in the proof-relevant setting, and this made efficient by a compact notation for globular pasting diagrams. We conclude with an application of our proof-relevant algebra framework to proof-relevant functor composition, and suggest further definitions in this vein.

Supervised by Emily Riehl

# Contents

# 1. Introduction

In recent years there has been growing interest in models of higher-category-like structures, and several theories have emerged each dealing with a different aspect of what higher-category-like might mean in a given domain. The most direct generalisation of ordinary categories to the infinite-dimensional setting is the model of strict $\omega$-categories. The definition is appealingly straightforward, but owing to the entirely strict and rigid nature of the formalism there are few if any interesting examples of strict $\omega$-categories (perhaps with the exception of the strict $\omega$-category of strict $\omega$-categories). In a different direction, the theory of quasi-categories originally introduced in [BV73] was developed to describe the composition of homotopy coherent natural transformations and has since proliferated to many other mathematical settings. In yet another direction, other authors have set out to capture a more algebraic notion of composition which is nevertheless appropriately weak. For instance, we are particular inspired by the work of [Lei09] which builds on and advances the idea that higher-category-like structures might have all of their composition operations determined at once by specifying the calculus of weak composition in the abstract and then defining higher categories to be those objects which model this abstract calculus of composition. The apparatus which renders possible this approach is that of algebras for contractible globular operads.

To match the proliferation of models of higher-category-like objects, important and productive work has been undertaken to make connections and effect comparisons between these models so as to make sure that we are at least all touching the same elephant—see for instance [Che03b; Che08; Ber18; Ver08]. Fundamentally we expect that the utility of a model of higher categories is measured by its intended application.

Our story therefore must begin by clarifying an intended application, and for us that will be the task of understanding composition not merely as a family of operations, but as something that inherently involves data. To be concrete we now outline three criteria which give meaning to this phrase, and will serve as more tangible standards to guide our endeavours and by which to measure our results.

### Composition is multi-valued

Already the insufficiency of single-valued composition and composition operations are visible at the level of monoidal categories. It is common to view monoidal categories as one-object bicategories so that the monoidal operation is thus composition. Many of the monoidal categories which appear in nature have their tensors given by a universal construction, examples abound: products, coproducts, modules, and so on. In this way tensor does not naturally produce one result, but many. Of course these results are all isomorphic, but nevertheless to produce a monoidal category requires that we choose from among them. It is common to assert that any structured choice will do but this leads us to ask: why make any choice at all?

There is a second important aspect to the idea that composition is multi-valued, and that is that it may therefore not be presented by a fixed set of composition operations. Let us work our way up to this stance in parts, beginning with the sense in which composition is biased in the classical definition of a monoidal category.

In the classical definition of a monoidal category there is a binary composition operation (the tensor) and a nullary composition operation (the unit). However this particular definition gives rise to an additional complication. Above and beyond being required to make some choice of result for each binary composite, we see that to compose $n$ objects we are required to make an additional choice in decomposing this operation into a sequence of binary composites. In order to alleviate this burden, we might look instead to adjust the *signature* of the definition: that portion which specifies the arities of the composition operations. In this way, for instance, we may derive the definition of an "unbiased" monoidal category. "Unbiased" monoidal categories specify not just a binary and nullary composition operation in their signatures, but a composition operation for each $n$. Of course care must be taken to add suitable axioms which relate all triple nestings of these new operations, but in principle the the definition proceeds in a similar fashion to the classical one. In an "unbiased" monoidal category, there is therefore a canonical of choice of composition operation to apply to every arrangement of $n$ objects. This idea of signature and "unbiased" extends by analogy to bicategories and similar algebraic structures, but from our perspective this approach does not solve any problems.

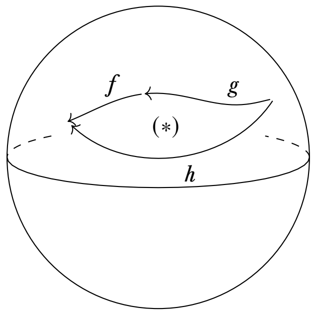
First let us note that the basic problem present in classical monoidal categories is still present in "unbiased" monoidal categories: despite there being now many composition operations, nevertheless a choice of result must be made for each one. But there is a deeper issue which is more fundamental. In the examples of monoidal categories in nature, those whose tensor is given by some universal property, the possible results of composition depend not only on *how many* things are being composed or in what order, but also on *precisely which* things are being composed. Therefore by fixing some signature for composition once and for all we will never be attentive to the true nature of composition, viz., that its potentially many results depend crucially on its arguments and not their shapes. For this reason we have been referring to "unbiased" with quotes: there is an inherent bias in predicating composition on the notion of signature at all.

This understanding informs our first criterion, but in order to state it neutrally we should adopt a slight change of language. Owing to the inherently multi-valued nature of composition and the fact that results depend sensitively on their inputs, it is best to distance ourselves from the language of fixed-signature composition operations and instead refer to *arrangements of cells* and *evaluations of arrangements*. In this way we take as primitive the idea that particular cells may be arranged into composable shapes as divorced from any notion of composition, and thereafter we allow for multiple evaluations of these arrangements and for the possibility of these evaluations to depend on the cells themselves. In this language our first criterion is as follows.

**Criterion 1.** A higher category must allow for many ways of evaluating a given arrangement of cells, and the results must be allowed to depend on the particular cells considered and not just the shape of their arrangement.

### Composition requires proof

The next facet of composition we wish to consider is that, in general, the process of composition itself entails data beyond merely the result. To see why this might be so, let us consider a topological example, namely the fundamental groupoid of a 2-sphere.




The presence of the patch marked (∗) informs us that, up to homotopy, we may regard $h$ as a composite of $f$ and $g$. However, there is another proof that $h$ is a composite, and that is given by the complement of the patch on the 2-sphere. The key point here is that these are fundamentally different proofs, and their difference—the lack of a higher homotopy between these—detects something genuinely interesting about the 2-sphere itself. In this way by disregarding the reason *why* we believe certain morphisms to be composites of others we risk losing important information about our category, in this case that our category is in fact a truncation of a higher category: the fundamental $\infty$-groupoid. This informs our second criterion by which we will judge potential models.

**Criterion 2.** A higher category must record the reason why a given cell is an evaluation of an arrangement of other cells.

### Proofs are not cells

Useful as the above example may be in understanding that the proofs of composition should carry data, it unfortunately confuses a subtle aspect of the situation. Note that in the fundamental $\infty$-groupoid many of the proofs of evaluation are in fact higher cells of that same category. It is tempting therefore to extrapolate that the form of evaluation proof data is always that of a higher cell in the category. Indeed, this the approach taken by many contemporary models of higher categories. For instance, we may seek to replace monoidal categories by *represented* multicategories: in a represented multicategory a tensor of a list of objects $A_1, \ldots, A_n$ is determined by a universal multimorphism whose domain is the list $\langle A_i \rangle$. Because there may in general be many universal multimorphisms with a fixed domain, a represented multicategory more naturally houses the multi-valued nature of evaluation than does a monoidal category.

However, this approach we argue is insufficient. In forcing the proof that a certain object is a tensor of other objects to be a morphism of the category we have, in many classes of examples, entirely lost the true proof. For instance, consider a multicategory representing a monoidal category whose tensor product is the categorical product. Then the real reason that an object $P$ is a product of the list $A_1, \ldots, A_n$ is that $P$ is the summit of a cone over the $A_i$ and moreover that this cone is proven to be terminal among all other cones over the same diagram. This information, the data of the cone legs and their accompanying universal property, is fundamental to proving that universal cells in the multicategory actually work, but it is entirely lost at the level of the multicategory: neither the cones or the proofs of their limiting nature are cells of the multicategory.[1]

The insufficiency of cells for recording the proofs of composition informs our final criterion.

**Criterion 3.** A higher category must allow for more general proofs of evaluation than cells of the category.

### Towards proof-relevant categories

We have thus expressed our criteria for higher-category-like structures, and briefly and with great bias noted that none of the standard approaches to monoidal categories are satisfactory. We have focused on monoidal categories for they are the lowest-dimensional examples where the challenges of composition are evident, but nevertheless monoidal categories are representative of the problems faced by higher-dimensional models. For instance, the operadic approach of [Bat98] and [Lei09, §9.2] constrains evaluation to be single-valued and fixes a signature, the opetopic approach of [BD98] allows multi-valued evaluation but requires that the proofs are cells of the category—a similar requirement to the complicial sets of [Ver06]. The question therefore arises: how shall we imagine a device which does allow us to study composition in the way we have outlined?

Our device ought to have a collection of cells, and if we are to take seriously our first criterion then we must afford it a primitive notion of arrangements of these cells into composable shapes. Let us be slightly more concrete and require that all of our cells are globular: they are of a finite dimension, and cells in dimension $n + 1$ have a single source globular cell and a single target globular cell, both of which are in dimension $n$. Thus our device has a globular set $A$ of cells.

Let us also decide that the primitive notion of arrangement here is that of a globular pasting: a collection of globular cells matched source-to-target in various dimen-

[1] We might say that to assert otherwise would be a category error.

sions in a composable shape. For globular sets $A$ the notion of globular pasting is controlled by the "free strict $\omega$-category monad" $T$, and so we have therefore specified that our device should have access to both a globular set $A$ of cells as well as to the globular set $TA$ of arrangements of these cells.

The second criterion informs us that this device must have access to a collection $P$ of proofs which match arrangements of cells to their evaluations. The third criterion informs us that these proofs must be allowed to be more general objects than cells, so that $P$ is data not contained in $A$. Putting these data together, we might therefore invent a device which looks something like a span of globular sets.

$$A \xleftarrow{c} P \xrightarrow{d} TA$$

A proof in $p \in P$ therefore records an evaluation $cp \in A$ and an arrangement $dp \in TA$, and $p$ is understood to prove that $dp$ evaluates to $cp$. Through this structure we have allowed ourselves potentially many proofs over a given arrangement, and therefore potentially many results of evaluation. We have demanded that all evaluations must be witnessed by proofs, and we have allowed proofs $p \in P$ to be of a different sort to the cells $A$. Unfortunately in this description we have not yet assured ourselves that every arrangement has an evaluation, or that there is any reasonable calculus of proofs—for instance, every cell should evaluate at least to itself. These separate aspects may be addressed by further machinery, and in total such a device is what we shall call a proof-relevant category.

### On designing models

Now that we have imagined a particular model, the question becomes one of implementation. How shall we make this idea of a proof-relevant category precise, and how in particular shall we situate it in a general theory of "like" objects?

To give an answer we shall adopt a somewhat algebraic approach: we shall define a calculus of compositions in the abstract and define proof-relevant categories to be those objects which model this calculus. As it turns out, this description alone does not uniquely determine our course for there is a further choice to be made. Common in models such as contractible operads is to have a sufficiently weak calculus of compositions so that those objects which *strictly* model this weak calculus of compositions are themselves weak. This is the approach taken by, for example, Leinster in the definition [Lei09, §9.2] of a higher category as a strict algebra for a certain operad $L$. However, as we have seen this determines once and for all a fixed signature for the allowable composition operations and so is not satisfactory.

Thus will adopt the complementary approach: we will deal only in a totally strict calculus of compositions, but provide here a novel notion of what it means for an object to model this calculus weakly. The rigorous embodiment of a calculus of compositions for us is that of a $T$-category—a many-object version of a globular operad

[Lei09, §4.2]. The notion of weak model we introduce here is that of proof-relevant algebra for a $T$-category. The notion of proof-relevant algebra allows each model to impose its own variations on the strict calculus of compositions and thereby we avoid specifying composition operations or a fixed signature.

By shifting the weakness to the notion of proof-relevant algebra and maintaining a strict notion of calculus, we obtain some further advantages and simplifications. First we do not have to contend with the intricate combinatorics of a weak calculus of composition. In contrast to the weak calculus of compositions $L$ of [Lei09, §9.2], our strict calculus of compositions is almost entirely straightforward—it has only a single abstract composition operation for each possible abstract arrangement. Second, the notion of proof-relevant algebra is sufficiently general that we do not have to restrict our attention to only those models of a calculus of composition in a single category. Rather the notion of proof-relevant algebra is developed relative to a given $T$-category. For instance, we may consider the $T$-category corresponding to the strict calculus of composition obtained between two categories and a functor between them. Proof-relevant algebras for this model something we call proof-relevant functors, and these have every indication of suitably generalising desirable aspects of other models weak functors, and of satisfying analogues of our criteria above for functors.

In fact, we will ultimately describe a way to take any strict $\omega$-category $\mathcal{C}$ to a $T$-category $\mathcal{SC}$ shaped by it: we will thus obtain the strict calculus of composition associated to replacing each object of $\mathcal{C}$ with a category, each 1-cell with a functor, and so on. Thus the strict calculus of compositions for a single category is $\mathcal{S}1$ where $1$ is the terminal $\omega$-category, and proof-relevant algebras for $\mathcal{S}1$ are proof-relevant categories. The strict calculus of compositions for a pair of categories and a functor is $\mathcal{S}2$ where $2 = \{\cdot \leftarrow \cdot\}$, and proof-relevant algebras for $\mathcal{S}2$ are proof-relevant functors. Proof-relevant algebras for $\mathcal{SC}$ are our central contribution, and this notion will allow us to suggest tentative definitions of a host of proof-relevant notions. In particular we study composites of proof-relevant functors as dictated by $\mathcal{S}3$, where $3$ is a commutative triangle.

## 1.1. Organisation of the work

In Section 2 we begin by recalling the necessary background on formal category theory, $T$-categories, globular sets, and strict $\omega$-categories. In Section 2.1 we recall some theorems and details about algebraic colimits of monads and in Section 2.4 we recall split contractions of globular morphisms, both of which see heavy use. In Section 2.5 we introduce our compact notation for globular pasting diagrams which permeates our work here, and renders efficient or sometimes even possible our arguments involving pasting diagrams.

Then in Section 3 we meet our model of higher-category-like structure, the proof-relevant category. There we expend substantial effort to understand the various features of these proof-relevant categories, and attempt to address the nature of identities and equivalences within them. To end our introduction to proof-relevant categories we explore a varied collection of examples: strict $n$- and $\omega$-categories, 1-category-like objects with un-truncated proof data, monoidal categories, bicategories, algebras for contractible globular operads, and proof-relevant categories where composition is determined by a universal property.

After this, in Section 4 we introduce an appropriate notion of morphism between proof-relevant categories, that of proof-relevant functors. We again put special emphasis on understanding the features of these devices, and especially concentrate on some of the perhaps more foreign subtleties they introduce: proof-relevant functors are not only unbiased in that they obtain values on entire arrangements of cells, but are also multi-valued. After investigating their interaction with equivalences and identities we turn to some examples.

Finally, in Section 5 we deal with the general framework of proof-relevant algebras we had alluded to above. We define what it means for a $T$-category to have a proof-relevant algebra in analogy with what it means to have a strict algebra. We prove various results about this new notion, including Theorem 5.2.1: despite being weak, proof-relevant algebras are still sufficiently structured enough to be finitarily monadic over graphs and in particular locally finitely presented as a category. We study various connections to algebras as a means of generating further examples, for instance we recover dependent sum (Lemma 5.3.2) and re-indexing (Lemma 5.3.3) as means of base change for proof-relevant algebras. Then in Section 5.4 we introduce the central device which enables the definitions of proof-relevant category and proof-relevant functor. As a consequence of Theorem B.3.15 of the appendix, we prove in Lemma 5.4.2 that there is a shape functor $\mathbb{S}\colon \omega\mathrm{Cat} \to T\text{-}\mathrm{Cat}$ along with a structured comparison between $\omega$-category algebras of $\mathcal{C}$ and proof-relevant algebras of $\mathbb{S}\mathcal{C}$. Using this shape functor, in Table 5.4.5 we suggest some further examples of strict bases whose proof-relevant algebras are of interest. As an application in Section 5.5 we study the notion of composition of proof-relevant functors that the shape functor suggests.

Attached to this work too are two appendices. The entirety of Appendix A is dedicated to the proof of Theorem 5.2.3, a crucial ingredient in our Theorem 5.2.1. In Appendix B we treat a topic of perhaps independent interest, viz., the confluence of internalisation for enriched categories, algebras for both enriched and internal categories, and the theory of $T$-categories. All of this is phrased through the language of virtual double categories, and we obtain Theorem B.3.15, a very general, enriched version of a Grothendieck construction. A particular case of this result underpins our shape functor of Section 5.4.

## 1.2. Acknowledgements

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*To a wonderful period of my life, and those who made it so.*

# 2. Background

In this section we cover the necessary background for this work. In Section 2.1 we recall some formal results about adjunctions, aspects of monadicity, and limits. Readers may be interested in a proof of a known result about algebraic colimits of monads which nevertheless seems to escape direct citations. In Section 2.2 we recall the relevant definitions and results of the theory of $T$-categories. In Section 2.3 we recall the notion of globular sets, and some properties of the category thereof. In Section 2.4 we recall the central notion of a split contraction of globular morphism and list some results pertaining to closure properties. Finally in Section 2.5 we recall the free strict $\omega$-category monad on globular sets, whose algebras are the strict $\omega$-categories. This monad defines the notion of globular pasting diagram, and we delineate a more compact notation for such pastings in the same section.

## 2.1. Adjunctions, colimits, and monadicity

In this section we recall some gentle lemmas about the interactions between pullbacks, adjunctions, colimits, and monadicity. In addition we give a proof for a theorem extractible from other sources about pullbacks of monadic finitary functors and locally finitely presentable categories.

**Definition 2.1.1.** A functor $F\colon \mathscr{C} \to \mathscr{D}$ is **finitary** if it preserves all filtered colimits. A monad $(T,\mu,\eta)$ on a category $\mathscr{C}$ is said to be **finitary** if its functor $T$ is finitary. ◃

**Lemma 2.1.2.** *The following interactions between comma categories and colimits hold.*

*(i) If $C \in \operatorname{ob}\mathscr{C}$ is an object of a category, then the projection functor $\mathscr{C}/C \to \mathscr{C}$ creates colimits.*

*(ii) If $\mathscr{A}$ and $\mathscr{B}$ are categories with filtered colimits, and $\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$ are finitary functors, then the comma category $(F \downarrow G)$ has filtered colimits and the projections $\mathscr{A} \leftarrow (F \downarrow G) \to \mathscr{B}$ are finitary.*

*Proof.* See, for instance, [Bor94a, Props. 2.16.3 & 2.16.1] respectively. ■

Recall that monadic adjunctions may be lifted to slices.

**Lemma 2.1.3.** *Fix a functor $R\colon \mathscr{C} \to \mathscr{D}$ and an object $X \in \operatorname{ob}\mathscr{C}$, and consider the induced functor $R'\colon \mathscr{C}/X \to \mathscr{D}/RX$. If $R$ is finitary then so too is $R'$. If $R$ has a left adjoint $L$ then $R'$ has a right adjoint $L'$, and moreover if the adjunction $L \dashv R$ is monadic then so too is the adjunction $L' \dashv R'$.*

*Proof.* To see that $R'$ is finitary when $R$ is, we argue as follows. In the below commutative square of functors, the vertical maps create colimits by Lemma 2.1.2 (i) above, and the bottom horizontal map preserves filtered colimits.

$$\begin{tikzcd} \mathscr{C}/X \arrow[r, "R'"] \arrow[d, "\pi"'] & \mathscr{D}/RX \arrow[d, "\pi"] \\ \mathscr{C} \arrow[r, "R"'] & \mathscr{D} \end{tikzcd}$$

Next, if $R$ has a left adjoint $L$, we may construct $L' \colon \mathscr{D}/RX \to \mathscr{C}/X$ as

$$L'(f\colon D \to RX) :\equiv LD \xrightarrow{Lf} LRX \xrightarrow{\varepsilon_X} X\ ,$$

on objects and extend in the evident manner to morphisms. A straightforward computation shows that we may take the components of the unit and counit of $L \dashv R$ as the components for the unit and counit for the putative adjunction $L' \dashv R'$. Finally, if $R$ is monadic we may appeal to the celebrated monadicity theorem of [Bec67] (or see, for example, [Bor94b, Thm. 4.4.4]) and deduce that $R'$ reflects isomorphisms and creates coequalisers for $R'$-split pairs because the same is true of $R$. ■

Under certain conditions, the all-but-adjoint monadicity conditions are stable under pullback.

**Lemma 2.1.4.** *Consider the below pullback square in* Cat *and suppose that $U$ is conservative and an isofibration, and that $\mathscr{D}$ has, and $U$ preserves, coequalisers of $U$-split pairs. Then $\pi_{\mathscr{B}}$ is conservative and an isofibration, and $\mathscr{D} \times_{\mathscr{C}} \mathscr{B}$ has, and $\pi_{\mathscr{B}}$ preserves, coequalisers of $\pi_{\mathscr{B}}$-split pairs. In particular the hypotheses on $U$ are satisfied if $U$ is monadic.*

$$\begin{tikzcd} \mathscr{D} \times_{\mathscr{C}} \mathscr{B} \arrow[r, "\pi_{\mathscr{B}}"] \arrow[d, "\pi_{\mathscr{D}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{B} \arrow[d, "P"] \\ \mathscr{D} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd}$$

*Proof.* Let us dispense with the last claim first: all monadic functors $U$ are conservative and isofibrations, and the category of algebras has, and $U$ preserves, coequalisers of $U$-split pairs.

Now let us address the conservativity of $\pi_{\mathscr{B}}$. Suppose that we have a morphism $(d,b)\colon (D,B) \to (D',B')$ in the pullback such that $b$ is an isomorphism in $\mathscr{B}$. Then in particular $Pb = Ud$ is an isomorphism between $PB = UD$ and $PB' = UD'$. As $U$ is conservative, $d$ must therefore be an isomorphism in $\mathscr{D}$ so that $(d,b)$ is an isomorphism in the pullback as desired. An entirely similar argument may be used to conclude that $\pi_{\mathscr{B}}$ is an isofibration.

Suppose next that for a parallel pair $(d,b),(d',b')\colon (D,B) \rightrightarrows (D',B')$ there is a split coequaliser on the parallel pair $b,b'\colon B \rightrightarrows B'$. Split coequalisers are preserved by all functors, and so in particular by $P$. But the image under $P$ of this split coequaliser is the structure of a $U$-split pair on $d,d'\colon D \rightrightarrows D'$ and $\mathscr{D}$ has, and $U$ preserves, coequalisers for $U$-split pairs. Thus there is a coequaliser of $d,d'\colon D \rightrightarrows D'$ whose image under $U$ agrees with the coequaliser on $Pb,Pb'\colon PB \rightrightarrows PB$ up to isomorphism. Because $U$ is an isofibration, we may re-choose the co-equaliser in $\mathscr{D}$ to agree under $U$ exactly with the coequaliser on $Pb,Pb'\colon PB \rightrightarrows PB$. Note that pairs of colimits indexed by the same category in $\mathscr{B}$ and $\mathscr{D}$ whose images under $U$ and $P$ are equal and colimits in $\mathscr{C}$, are colimits in the pullback. Thus we conclude that $\mathscr{D} \times_{\mathscr{C}} \mathscr{B}$ has, and $\pi_{\mathscr{B}}$ preserves, coequalisers of $\pi_{\mathscr{B}}$-split pairs. ■

The various ingredients of the following theorem may be seen as consequences of [Lei09, Thm. G.1.1] which draws on the work [Kel80, Thm. 27.2], and of general aspects of the theory of locally presentable categories [AR94, §2.H], [MP89, §5.1] and the theory of algebras for 2-monads [BKJ89] (or see for instance [Lac10, §6]). Nevertheless, we attempt to give a somewhat elaborated account of the result.

**Theorem 2.1.5.** *Given the below pullback in* Cat*, if $U$ and $U'$ are both finitary and monadic, and $\mathscr{D}$ is locally finitely presentable, then all the categories in the square are locally finitely presentable, and $P$, $Q$, and $UP = U'Q$ are monadic and finitary.*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{B} \arrow[d, "U"] \\ \mathscr{C} \arrow[r, "U'"'] & \mathscr{D} \end{tikzcd}$$

The intended audience for this proof are those who are comfortable deferring to certain facts about locally finitely presentable categories, but not comfortable deferring to and specialising the very general arguments of [Kel80].

*Proof.* The strategy here will be to demonstrate that all the functors concerned are finitary, continuous and satisfy the all-but-adjoint monadicity criteria. Then as all the categories will have been seen to be locally finitely presentable, these functors will have left adjoints and hence will be monadic as claimed.

By, for example [AR94, Rem. 2.75], because $U$ and $U'$ are finitary and monadic, and $\mathscr{D}$ is locally finitely presentable, the categories $\mathscr{C}$ and $\mathscr{B}$ are locally finitely presentable too. Moreover, because monadic functors are in particular isofibrations as may be readily verified, we may verify that all limits and colimits $\mathscr{A}$ are created jointly by $P$ and $Q$ over $\mathscr{D}$ and the pullback is a pseudo-limit in Cat. Furthermore, although it is difficult to find a direct reference other than [Bir84, Thm. 2.17][2], $\mathscr{A}$ is locally finitely presentable.

Already then, $P$, $Q$ and the diagonal $UP = U'Q$ are finitary and continuous. In addition, we now have enough to determine that both $P$, $Q$, and $UP = U'Q$ satisfy the all-but-adjoint monadicity criteria. We will argue only explicitly in the case of $P$, as $U$ is similar and the general proof strategy for the diagonal is the same. To that end, we want to establish that $P$ is conservative and that $\mathscr{A}$ has, and $P$ preserves, coequalisers of $P$-split pairs. Note that we meet the hypotheses of Lemma 2.1.4, and so the claim follows.

To conclude the proof we need only establish that $P$, $Q$ and the diagonal have left adjoints. However, they are all finitary and continuous, and so by [Bor94b, Thm. 5.5.7] for example, they each have left adjoints and are therefore monadic. ■

[2]See also the discussion after [MP89, Prop. 5.1.7] or the content of [AR94, Exercises 2.l, 2.m, & 2.n].

## 2.2. Generalised $T$-categories

We will now briefly remind the reader of relevant elements of the theory of [Lei09]. Our interest in this matter will almost exclusively be constrained to the case when the category in question is the category of globular sets of Section 2.3 and the monad is the free strict $\omega$-category monad of Section 2.5, nevertheless we reproduce below some aspects of the general theory.

**Definition 2.2.1.** A monad $(T,\mu,\eta)$ on a category $\mathscr{C}$ is **cartesian** when $\mathscr{C}$ has and $T$ preserves pullbacks, and the naturality squares of $\mu$ and $\eta$ are cartesian. ◃

**Definition 2.2.2.** The classical bicategory $\mathrm{Span}(\mathscr{C},T)$ has as objects $\mathrm{ob}\,\mathscr{C}$, as 1-cells $X \to Y$ spans $X \leftarrow Z \to TY$ in $\mathscr{C}$, as 2-cells span morphisms, and composition and related data is derived from $\eta$, $\mu$, $T$ and pullback. ◃

*Remark* 2.2.3. Note that although this classical bicategory does enjoy weak associativity, owing to the cartesian nature of $\eta$, we will consider 1-cell composition with identities to be strict. That is, we consider pullback along the identity morphism to extend to the identity functor. ◃

**Definition 2.2.4.** Let $T$ be a cartesian monad on a category $\mathscr{C}$. The **category** $T$-Grph **of** $T$**-graphs** is defined to be the category whose objects are endo-1-cells of $\mathrm{Span}(\mathscr{C},T)$ and whose morphisms $(f^0,f^1)\colon (A^0,A^1,A^\ell,A^r) \to (B^0,B^1,B^\ell,B^r)$ are pairs of morphisms of $\mathscr{C}$, $(f^0\colon A^0 \to B^0, f^1\colon A^1 \to B^1)$, which render commutative the following diagram.

$$\begin{tikzcd}
A^0 \arrow[d, "f^0"'] & A^1 \arrow[l, "A^\ell"'] \arrow[r, "A^r"] \arrow[d, "f^1"] & TA^0 \arrow[d, "Tf^0"] \\
B^0 & B^1 \arrow[l, "B^\ell"] \arrow[r, "B^r"'] & TB^0
\end{tikzcd}$$

Note that when $\mathscr{C}$ has products, $T$-Grph is also the comma category $(\mathscr{C} \downarrow F)$ where $F\colon \mathscr{C} \to \mathscr{C}$ is the functor $FX :\equiv X \times TX$. ◃

**Definition 2.2.5.** Let $T$ be a cartesian monad on a cartesian category $\mathscr{C}$. A $T$-**category** is a monad in $\mathrm{Span}(\mathscr{C},T)$, and a **morphism of** $T$**-categories**

$$(F^0,F^1)\colon (C^0 \leftarrow C^1 \to TC^0) \to (D^0 \leftarrow D^1 \to TD^0)$$

is the data of a pair $(F^0\colon C^0 \to D^0,\ F^1\colon C^1 \to D^1)$ of maps of $\mathscr{C}$ which commute with the monad structure in the evident sense. Each $T$-category has an underlying $T$-graph, and similarly so for morphisms. These observations give rise to the forgetful functor $U\colon T\text{-Cat} \to T\text{-Grph}$. ◃

More explicitly, a $T$-category comprises the data of an object $C^0$ of $\mathscr{C}$, a span $C^0 \xleftarrow{d} C^1 \xrightarrow{c} TC^0$ in $\mathscr{C}$, and morphisms $\mathsf{refl}\colon C^0 \to C^1$ and $\mathsf{comp}\colon C^1 \circ C^1 \to C^0$ in $\mathscr{C}$. Together these morphisms satisfy the monad laws in $\mathrm{Span}(\mathscr{C},T)$: associativity of comp and unitality of comp and refl.

When no confusion is possible we will choose to abbreviate the data and properties of a $T$-category by double-struck letters representative of its underlying object. That is, we shall write $\mathbb{C}$ in place of $(C^0, C^1, d, c, \mathsf{refl}, \mathsf{comp})$ + (monad laws).

*Remark* 2.2.6. $T$-graphs and $T$-categories are connected to a broader framework. When $\mathscr{C}$ is cartesian, and in the terminology of [CS09], one may construct the horizontal Kleisli virtual double category $\mathbb{H}\text{-Kl}(\mathbb{S}\text{pan}\,\mathscr{C}, T)$ of [CS09, Def. 4.1], as well as the virtual double category of modules $\mathbb{K}\mathrm{Mod}(\mathscr{C}, T) :\equiv \mathbb{M}\mathrm{od}(\mathbb{H}\text{-Kl}(\mathbb{S}\text{pan}\,\mathscr{C}, T))$ therein [CS09, Defs. 2.8 & 4.3]. In this language, our $T$-graphs appear as horizontal endomorphisms in $\mathbb{H}\text{-Kl}(\mathbb{S}\text{pan}\,\mathscr{C}, T)$ and the category $T$-Grph may be appropriately derived from this. Additionally, our category $T$-Cat is the vertical category of $\mathbb{M}\mathrm{od}(\mathbb{H}\text{-Kl}(\mathbb{S}\text{pan}\,\mathscr{C}, T))$. ◃

Leinster in his applications is interested in studying when the evident forgetful functor $T\text{-Cat} \to T\text{-Grph}$ is monadic. For this we need some terminology.

**Definition 2.2.7** ([Lei09, App. D]). A diagram $\omega \to \mathscr{C}$ is said to be a **nested sequence** if the image of every arrow $n \to n{+}1$ of $\omega$ in $\mathscr{C}$ is monic. With this terminology, let us say that a **category $\mathscr{C}$ is suitable** if

(i) $\mathscr{C}$ is cartesian,

(ii) $\mathscr{C}$ has disjoint finite coproducts which are stable under pullback, and

(iii) $\mathscr{C}$ has colimits of nested sequences, and these colimits commute with pullbacks and have monic coprojections

**A monad $(T, \mu, \eta)$ is suitable** if it is cartesian and if $T$ preserves colimits of nested sequences. ◃

**Theorem 2.2.8** ([Lei09, Thms. 6.5.1, 6.5.2 & Prop 6.5.6]). *Let $T$ be a suitable monad on a suitable category $\mathscr{C}$. Then the forgetful functor $U\colon T\text{-Cat} \to T\text{-Grph}$ has a left adjoint, the adjunction is monadic, and both $\mathscr{C}^+ :\equiv T\text{-Grph}$ and the induced monad $T^+$ on $\mathscr{C}^+$ are suitable.*

*Any presheaf category is suitable, and any finitary monad on a cartesian category is suitable. In addition, if $\mathscr{C}$ is a presheaf category and the functor $T$ preserves wide pullbacks then the same is true of $\mathscr{C}^+$ and $T^+$. Finally, if $T$ is finitary then so is $T^+$.*

For the remainder of this section let us work in the context of a fixed cartesian monad $T$ on a cartesian category $\mathscr{C}$. We conclude this section by recalling the notion of algebras for a $T$-category as well as the accompanying notion of "generalised category of elements" which sends such algebras to $T$-categories of their own.

**Lemma 2.2.9** ([Lei09, §4.3]). *There is a pseudo-functor $T_{(-)}\colon T\text{-Cat}^{\mathrm{op}} \to \mathrm{Mnd}(\mathrm{Cat})$ where the domain is considered as a discrete 2-category, sending $T$-categories $\mathbb{C}$ to monads $T_{\mathbb{C}}$ on $\mathscr{C}/C^0$ and morphisms $F\colon \mathbb{C} \to \mathbb{D}$ of $T$-categories to lax monad morphisms $T_F\colon T_{\mathbb{D}} \to T_{\mathbb{C}}$.*

*Proof.* We will begin by describing a functor $T\text{-Cat}^{\mathrm{op}} \to \mathrm{EndoFunc}$ which sends $T$-categories and their morphisms to endofunctors and lax morphisms, before verifying that this functor factors through the forgetful functor $\mathrm{Mnd}(\mathrm{Cat}) \to \mathrm{EndoFunc}$.

On objects, a $T$-category $\mathbb{C}$ is sent to the composite of functors

$$\mathscr{C}/C^0 \xrightarrow{T} \mathscr{C}/TC^0 \xrightarrow{d^*} \mathscr{C}/C^1 \xrightarrow{\Sigma_c} \mathscr{C}/C^0 \ ,$$

and on morphisms $F\colon \mathbb{C} \to \mathbb{D}$ of $T$-categories to the following lax morphism of endofunctors

$$\begin{tikzcd}
\mathscr{C}/D^0 \arrow[r, "T"] \arrow[d, "(F^0)^*"'] \arrow[dr, phantom, "\cong"] & \mathscr{C}/TD^0 \arrow[r, "d^*"] \arrow[d, "(TF^0)^*"] \arrow[dr, phantom, "\overset{\hat{F}}{\Rightarrow}"] & \mathscr{C}/D^1 \arrow[r, "\Sigma_c"] & \mathscr{C}/D^0 \arrow[d, "(F^0)^*"] \\
\mathscr{C}/C^0 \arrow[r, "T"'] & \mathscr{C}/TC^0 \arrow[r, "d^*"'] & \mathscr{C}/C^1 \arrow[r, "\Sigma_c"'] & \mathscr{C}/C^0
\end{tikzcd} ,$$

where the natural transformation $\hat{F}$ arises from the data of $F$ and the universal property of pullback. It may be verified that the the assignment outlined here is a pseudofunctor $T\text{-Cat}^{\mathrm{op}} \to \mathrm{EndoFunc}$. Let us pause to record here the elementary constituents of this construction so far. If $\mathbb{C}$ is a $T$-category and $p\colon X \to C^0$ is an object of $\mathscr{C}/C^0$, then the endofunctor $T_{\mathbb{C}}$ on $p$ may be seen to be $T_{\mathbb{C}}p \equiv c \circ d^*(Tp)$, where $d^*(Tp)$ is given by the below-left pullback. On morphisms $f\colon p \to p'$, $T_{\mathbb{C}}(f)$ is given by pullback factorisation, as rendered below-right.

$$\begin{tikzcd}
T_{\mathbb{C}}X \arrow[r, "d^*(Tp)"] \arrow[d, "q_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^1 \arrow[r, "c"] \arrow[d, "d"] & C^0 \\
TX \arrow[r, "Tp"'] & TC^0
\end{tikzcd}
\qquad
\begin{tikzcd}
T_{\mathbb{C}}X \arrow[r, dashed, "T_{\mathbb{C}}f"] \arrow[d, "q_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & T_{\mathbb{C}}X' \arrow[r, "d^*(Tp')"] \arrow[d, "q_{X'}"] \arrow[dr, phantom, "\lrcorner", very near start] & C^1 \arrow[d, "d"] \\
TX \arrow[r, "Tf"'] & TX' \arrow[r, "Tp'"'] & TC^0
\end{tikzcd}$$

This endofunctor $T_{\mathbb{C}}$ is equipped with a canonical monad structure inherited from the structure of the $T$-category $(C^0, C^1, d, c, \mathsf{refl}, \mathsf{comp})$, where $\eta_p :\equiv \langle \mathsf{refl}\, p, \eta^T_X \rangle$ and $\mu_p$ arise as indicated in the second pair of diagrams.

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[dr, dashed, "\eta_X"] \arrow[ddr, bend right, "\eta^T_p"'] & C^0 \arrow[r, "\mathsf{refl}"] & C^1 \arrow[d, equal, phantom] \\
& T_{\mathbb{C}}X \arrow[r, "d^*(Tp)"] \arrow[d, "q_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^1 \arrow[d, "d"] \\
& TX \arrow[r, "Tp"'] & TC^0
\end{tikzcd}
\qquad
\begin{tikzcd}
(T_{\mathbb{C}})^2X \arrow[r] \arrow[d, "q_{T_{\mathbb{C}}X}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[drr, dashed, bend left, "\mu_p"] & C^1 \circ C^1 \arrow[d] \arrow[drr, bend left, "\mathsf{comp}"] & & \\
TT_{\mathbb{C}}X \arrow[r] \arrow[d, "Tq_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & TC^1 \arrow[d] & T_{\mathbb{C}}X \arrow[r, "d^*(Tp)"] \arrow[d, "q_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^1 \arrow[d, "d"] \\
T^2X \arrow[r, "T^2p"'] \arrow[rr, bend right, "\mu^T_X"'] & T^2C^0 & TX \arrow[r, "Tp"'] & TC^0
\end{tikzcd}$$

To conclude the proof it remains to verify that $T_F$ is a lax morphism of monads whenever $F$ is a morphism of $T$-categories. Inevitably this amounts to the fact that $F$ respects the structure of the $T$-categories at issue, and that structure is precisely the structure of the monads $T_{\mathbb{C}}$. ■

With the construction of $T_{\mathbb{C}}$ achieved, we are ready to consider the category of algebras for the monad $T_{\mathbb{C}}$ induced by a $T$-category $\mathbb{C}$. By combining Lemma 2.2.9

with the usual Eilenberg-Moore construction observe that we have obtained a functor $T\text{-Cat}^{\mathrm{op}} \to \mathrm{Cat}$ which maps $\mathbb{C} \mapsto (\mathscr{C}/C^0)^{T_{\mathbb{C}}}$.

As we might expect from such a theory however, there is a good comparison between algebras of the monad $T_{\mathbb{C}}$ induced by a $T$-category $\mathbb{C}$ and $T$-categories fibred over $\mathbb{C}$—see [Lei09, Theorem 6.3.1], also recalled below. To state this we need some additional standard terminology.

**Definition 2.2.10.** Let $\mathbb{C}$ be a T-category, then a **discrete opfibration over** $\mathbb{C}$ is a morphism $F\colon \mathbb{D} \to \mathbb{C}$ of $T$-categories whose domain square

$$\begin{tikzcd} D^1 \arrow[r, "d_{\mathbb{D}}"] \arrow[d, "F_1"'] & TD^0 \arrow[d, "TF_0"] \\ C^1 \arrow[r, "d_{\mathbb{C}}"'] & TC^0 \end{tikzcd}$$

is a pullback. The **category of discrete opfibrations over** $\mathbb{C}$, denoted $\mathrm{DOpFib}(\mathbb{C})$, is defined to be the full subcategory of $T\text{-Cat}/\mathbb{C}$ whose objects are discrete opfibrations. There is an evident fully-faithful forgetful functor $\mathrm{DOpFib}(\mathbb{C}) \to T\text{-Cat}/\mathbb{C}$.

By pullback along morphisms of $T$-categories, the assignment $\mathbb{C} \mapsto \mathrm{Alg}\,\mathbb{C}$ extends to the **discrete opfibration pseudo-functor** $\mathrm{Alg}\colon T\text{-Cat}^{\mathrm{op}} \to \mathrm{Cat}$ where understand $T\text{-Cat}^{\mathrm{op}}$ to be a discrete 2-category. ◁

**Theorem 2.2.11** ([Lei09, Thm. 6.3.1]). *There is a "generalised category of elements" functor* $\mathrm{el}_{\mathbb{C}}\colon (\mathscr{C}/C^0)^{T_{\mathbb{C}}} \to \mathrm{DOpFib}(\mathbb{C})$*, which is an equivalence for every $T$-category* $\mathbb{C}$.

*Proof.* Let $(p\colon X \to C, h)$ be an algebra of $T_{\mathbb{C}}$ for a $T$-category $\mathbb{C}$. The underlying $T$-graph $(X, T_{\mathbb{C}}X, h, q_X)$ of the $T$-category $\mathrm{el}_{\mathbb{C}}(p,h)$ is constructed as indicated in the diagram below, and it is a straightforward computation to verify that $\eta_p$ and $\mu_p$ equip this graph with the structure of a $T$-category.

$$\begin{tikzcd} X \arrow[d, "p"'] & T_{\mathbb{C}}X \arrow[l, "h"'] \arrow[r, "q_X"] \arrow[d, "d^*(Tp)"] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "Tp"] \\ C^0 & C^1 \arrow[l, "c"] \arrow[r, "d"'] & TC^0 \end{tikzcd}$$

We extend this assignment of objects to one of morphisms by the functoriality of $T_{\mathbb{C}}$ and $T$ and it may be verified that $\mathrm{el}_{\mathbb{C}}\, f$ is a morphism of $T$-categories. From here it is a straightforward computation to check that $\mathrm{el}_{\mathbb{C}}$ is a fully-faithful functor, and to recognise that it is essentially surjective. ■

This result will allow us to freely confuse algebras for the monad $T_{\mathbb{C}}$ induced by a $T$-category $\mathbb{C}$ and discrete opfibrations over $\mathbb{C}$. In fact, for this reason we make the following definition.

**Definition 2.2.12.** The **category of algebras and algebra morphisms for a $T$-category** $\mathbb{C}$, written $\mathrm{Alg}\,\mathbb{C}$, is defined to be the category $\mathrm{DOpFib}\,\mathbb{C}$ of discrete opfibrations over $\mathbb{C}$. ◁

In the special case of $\mathbb{C} = 1$, the terminal $T$-category, we see that we have in fact constructed an embedding of the category of algebras of $T$.

**Lemma 2.2.13.** $\mathrm{el}_1 \colon \mathrm{Alg}(1) \to T\text{-Cat}$ *is a fully-faithful functor.* ∎

## 2.3. Globular sets

In this section we briefly recall a few aspects of the category of globular sets, a category central to many of the undertakings and definitions we give. Once we have settled this definition, in the coming sections we will recall the core notions of split contractions and strict $\omega$-categories.

**Definition 2.3.1.** The **globe category**, $\mathbb{G}$, is defined as having objects $\mathrm{ob}\,\mathbb{G} :\equiv \mathbb{N}$ and morphisms generated by $\mathbb{G}(n, n+1) :\equiv \{\sigma_n, \tau_n\}$ subject to the **globe relations**

$$\begin{aligned} \sigma_{n+1}\sigma_n &= \tau_{n+1}\sigma_n \\ \sigma_{n+1}\tau_n &= \tau_{n+1}\tau_n \end{aligned} \quad .$$

The category of **globular[3] sets** is $\widehat{\mathbb{G}} :\equiv [\mathbb{G}^{\mathrm{op}}, \mathrm{Set}]$. If $X$ is a globular set then we will refer to $x \in X_m$ as an $m$-**dimensional cell**. ◁

Note well that our category of globes is what some authors term "irreflexive", there are no morphisms $n+1 \to n$.

*Remark* 2.3.2. If the generators and relations presentation of $\mathbb{G}$ is troubling to the reader then we may leverage the specifics of the situation to reformulate $\mathbb{G}$ as the category on objects $\mathbb{N}$ with morphism sets given by

$$\mathbb{G}(n, m) :\equiv \begin{cases} \emptyset, & n > m \\ \{*\}, & n = m \\ \{0, 1\}, & n < m \end{cases} \quad .$$

Composition $\mathbb{G}(m,k) \times \mathbb{G}(n,m) \to \mathbb{G}(n,k)$ is defined by cases, and in the only non-trivial case $(n < m < k)$ composition is given by the second projection. ◁

One of the early properties that we wish to highlight of the category of globular sets is the following.

**Lemma 2.3.3.** $\widehat{\mathbb{G}}$ *is an infinitary extensive category. In elementary terms,* $\widehat{\mathbb{G}}$ *has all small coproducts and pullbacks of arbitrary coproduct inclusions exist. Moreover, for any family of commutative squares*

$$\begin{tikzcd} X_i \arrow[r, "\gamma_i"] \arrow[d, "f_i"'] & Z \arrow[d, "g"] \\ Y_i \arrow[r, "\iota_i"'] & \coprod_I W_i \end{tikzcd} ,$$

[3] globe-u-lar

*the top family of morphisms $\{\gamma_i\}$ are coproduct inclusions if and only if all the squares are pullbacks.*

*Proof.* This result holds at the level of presheaf categories, because limits and colimits are computed coordinate-wise in Set. As the reader may elect to verify, Set is infinitary extensive. ■

Next we are interested in isolating the notion of a free $n$-cell and its boundary.

**Definition 2.3.4.** The **free $n$-cell**, written $2_n$, is the globular set given by the representable functor of $\widehat{\mathbb{G}}$ on the object $n \in \mathrm{ob}\,\mathbb{G}$. That is, the globular set $2_n$ is given in dimension $m \in \mathbb{N}$ by the formula for $\mathbb{G}(m,n)$ above (note the switch). The **boundary of the free $n$-cell**, written $\partial_n$, is a sub-globular set $\partial_n \rightarrowtail 2_n$ which we shall now define by induction. In the base case, we define $\partial_0 :\equiv \emptyset$ with the evident inclusion. For the successor case, $\partial_{n+1}$ is defined as the following pushout, and the inclusion is defined via the copairing $[y\sigma, y\tau]$.

$$\begin{tikzcd}
\partial_n \arrow[r, tail] \arrow[d, tail] & 2_n \arrow[d, tail] \\
2_n \arrow[r, tail] & \partial_{n+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Note that by the Yoneda lemma, the free $n$-cell (co)classifies $n$-cells in globular sets: $\widehat{\mathbb{G}}(2_n, X) \cong X_n$. ◃

*Remark* 2.3.5. We may compute the globular set $\partial_n$ to be given in dimension $m \in \mathbb{N}$ by the formula

$$(\partial_n)_m :\equiv \begin{cases} \emptyset, & m > n \text{ or } n = 0 \\ \{0,1\}, & m \leq n \text{ and } n > 0 \end{cases} .$$

By induction we see that there is a canonical inclusion $\partial_n \rightarrowtail 2_{n+k}$ for each $k \geq 0$, and that these inclusions are coherent with respect to the inclusions $\partial_n \rightarrowtail \partial_{n+1}$. ◃

To conclude this section let us record that both the free $n$-cell and its boundary are finitely presented objects in $\widehat{\mathbb{G}}$.

**Lemma 2.3.6.** *The category $\widehat{\mathbb{G}}$ of globular sets is locally finitely presentable, and the finitely presented objects are generated under finite colimits by the representable functors. In particular, both $2_n$ and $\partial_n$ are finitely presented.*

*Proof.* A reference for the fact $\widehat{\mathbb{G}}$—presheaves on a small category—is locally finitely presentable and the fact that the finitely presented objects are generated by the representable functors is [Bor94b, Example 5.2.2(b)]. The free $n$-cell is a representable, and its boundary is inductively a finite colimit of other finitely presented objects. ■

## 2.4. Split contractions of globular sets

In [Lei09, §9.1], and the earlier works by the same author from which this is drawn, Leinster introduces the notion of a contraction on a map of globular sets. There

(Definition 9.1.2 *ibid.*) and elsewhere in these earlier works, the definition appears in elementary terms about parallel cells in a globular set. This elementary definition admits a natural rephrasing in terms of the notion of solutions to lifting problems, and we now give a definition along these lines under a related name.

**Definition 2.4.1.** Given a globular morphism $q\colon X \to Y$, the data of a **split contraction** $\kappa$ on $q$ is an assignment of solutions to all lifting problems of the form

$$\begin{tikzcd} \partial_n \arrow[r, "\partial x"] \arrow[d, tail] & X \arrow[d, "q"] \\ 2_n \arrow[r, "y"'] \arrow[ur, dashed, "\kappa(y,\partial x)" description] & Y \end{tikzcd} .$$

We write $q\colon X \overset{\sim}{\twoheadrightarrow} Y$ to indicate the pair of the globular morphism $q$ together with the data of a split contraction on $q$. A morphism $(f,g)\colon q \to q'$ in the arrow category $\mathrm{Ar}\,\widehat{\mathbb{G}}$ between split contractions $q\colon X \overset{\sim}{\twoheadrightarrow} Y$ and $q'\colon X' \overset{\sim}{\twoheadrightarrow} Y'$ is said to be **morphism of split contractions** when the solutions to all lifting problems are compatible: $\kappa'(gy, f\partial x) = f\kappa(y,\partial x)$. This gives rise to the **category** $\mathrm{SCon}\,\widehat{\mathbb{G}}$ **of split contractions** and the evident functor $\mathrm{SCon}\,\widehat{\mathbb{G}} \to \mathrm{Ar}\,\widehat{\mathbb{G}}$. ◃

In general, a split contraction on a globular map is *data* – there are many, different solutions to a given lifting problem. With this in mind, when we later refer to the lemmas below which equip various morphisms with the data of contractions we mean for the reader to understand that we may in fact be referring to the *particular* constructions here exhibited.

*Remark* 2.4.2. The lifting problems are qualitatively different in dimensions $n = 0$ and $n > 0$. In the first case, observe that $\partial_0 = \emptyset$ and so that the lifting problem degenerates to giving a section $s_0\colon Y_0 \to X_0$ of the set function $q_0\colon X_0 \to y_0$. In dimension $n > 0$, the lifting problems are non-degenerate and their solution lends to $q$ the structure of what [Lei09, §9.3] terms a contraction.

As Leinster explains, a contraction is designed to capture "weakly injective" maps in some directed sense: a cell in the base above a fixed boundary may be understood as a directed witness of "sameness", and a contraction allows this witness to be lifted to a specified boundary in the fibre. We wish to comment that the same phenomenon may also be understood as a form of "local surjectivity": whenever there is a cell in the base whose boundary is in the image of the map, there is a cell in the fibre which allows us to continue the surjectivity "upward" in dimensions. ◃

On this account, it is reasonable to expect, and indeed it is the case, that the section of a split contraction in dimension zero may be extended to a genuine section in $\widehat{\mathbb{G}}$.

**Lemma 2.4.3.** *Let $q\colon X \overset{\sim}{\twoheadrightarrow} Y$ be equipped with the data of a split contraction, then there is an extension of the section $s_0$ of $q_0$ to a section $s$ of $q$ in $\widehat{\mathbb{G}}$.*

*Proof.* We argue by induction on dimension. The base $n = 0$ is already given, so suppose that we have a section $s_k$ of $q_k$ for all $k \leq n$ such the collection $\{s_k\}_{k \leq n}$ satisfies the globular identities. To extend the section to a map $s_{n+1}$, given a cell in $y \in Y_{n+1}$ consider its boundary $\partial y \in Y_n$, apply the section $s_n$ to find a boundary $\partial x \in X_n$ in the correct fibre, and finally lift $y$ over $\partial x$. ■

Note that we do not wish to imply that split contractions are a form of weak equivalences. Indeed, it is not generally the case that injectivity and surjectivity make for bijectivity. More so here where the possible "weak injectivity" is manifest only in the higher cells of the globular sets, so that in their absence 2-of-3 certainly fails for split contractions. Along these lines let us now make contact with the theory of lifting problems.

We recall now some standard results concerning morphisms defined by right lifting structures. First, the category of split contractions $\mathrm{SCon}\,\widehat{\mathbb{G}}$ defines the category of right maps for a cofibrantly generated algebraic factorisation systems (AWFS). In particular, drawing on this theory allows us to prove the following.

**Lemma 2.4.4.** *The forgetful functor* $\mathrm{SCon}\,\widehat{\mathbb{G}} \to \mathrm{Ar}\,\widehat{\mathbb{G}}$ *is finitary and monadic, and* $\mathrm{Ar}\,\widehat{\mathbb{G}}$ *is locally finitely presentable.*

*Proof.* Consider the small discrete category $\mathrm{ob}\,\mathcal{J} = \{\partial_n \rightarrowtail 2_n\}$ whose objects are boundary inclusions in all dimensions, and note that $\mathrm{Ar}\,\widehat{\mathbb{G}} \cong \mathrm{Psh}(2 \times \mathbb{G})$ is a presheaf category and so in particular locally finitely presentable (see for example [Bor94b, Example 5.2.2(b)]). Thus the evident functor $\mathcal{J} \to \mathrm{Ar}\,\widehat{\mathbb{G}}$ meets the criteria of [BG16, Prop. 16], and there is an AWFS $(L, R)$ on $\mathrm{Ar}\,\widehat{\mathbb{G}}$ cofibrantly generated by $\mathcal{J}$, whose underlying monad $R$ has as algebras the split contractions, $\mathrm{SCon}\,\widehat{\mathbb{G}} \simeq (\mathrm{Ar}\,\widehat{\mathbb{G}})^R$. Note that the claim that $R$ is finitary is not an explicit conclusion of the stated proposition. However, the proof *ibid.* shows that the index of accessibility of the monad $R$ is the same as the least $\kappa$ such that all of the domains and codomains of $\{Uj\}_{j \in \mathrm{ob}\,\mathcal{J}}$ are $\kappa$-presented. In our case, $2_n$ is a representable and $\partial_n$ is a finite colimit thereof, so by Lemma 2.3.6 we have $\kappa = \aleph_0$ and $R$ is finitary. ■

The theory of AWFS ensures furthermore that we have a rich stock of split contractions in $\mathbb{G}$, and we have the following standard results.

**Lemma 2.4.5.** *Split contractions enjoy the following closure properties.*

*(i)* *Let* $q\colon X \xrightarrow{\cong} Y$ *be an isomorphism, then the data of a split contraction on* $q$ *is uniquely determined.*

*(ii)* *Let* $q\colon X \overset{\sim}{\twoheadrightarrow} Y$ *and* $r\colon Y \overset{\sim}{\twoheadrightarrow} Z$ *be equipped with the data of split contractions, then there are the data of a split contraction on the composite* $rq\colon X \to Z$.

*(iii)* *In the following pullback*

$$
\begin{tikzcd}
f^*X \arrow[r] \arrow[d, "f^*q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "q"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}
,
$$

*if $q\colon X \xrightarrow{\sim}\!\!\!\!\rightarrow Y$ is equipped with the data of a split contraction, then there are the data of a split contraction on $f^*q\colon f^*X \to Z$ such that the above square is a morphism of split contractions.*

■

There are further facets beyond these and we invite the inexperienced reader to begin their exciting exploration of the world of AWFS by attempting to descry yet additional structures and properties formally derivable for the category of right morphisms.

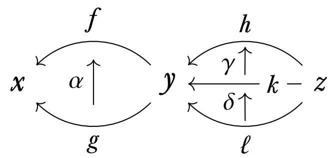
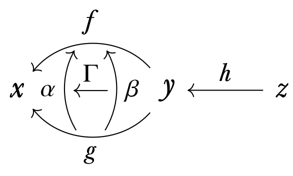
## 2.5. Strict $\omega$-categories

For a lucid account of the subject of this section we direct the reader to [Lei09, §8.1]. We wish only to highlight certain aspects of the theory of strict $\omega$-categories inasmuch as it meets the category of globular sets.

**Theorem 2.5.1** ([Lei09, Thm F.22 & Prop. C.3.2]). *The forgetful functor $\omega$-Cat $\to \widehat{\mathbb{G}}$ is monadic, and the induced **free strict $\omega$-category monad** monad is finitary, cartesian, coproduct preserving and preserves wide pullbacks.*

The free strict $\omega$-category monad $T$ takes a globular set $X$ to its globular set of pasting diagrams $TX$. By this terminology we mean all those formally composible arrangements of cells of $X$ which include in particular lower-dimensional cells considered as degenerate higher dimensional cells. As our globular sets are irreflexive there can be no Eckman-Hilton-style subtleties, and every possible formally composible arrangement of cells unambiguously determines a cell of $TX$, and visa versa. For a further treatment of this and the monad, see [Lei09, App. F].

**Notation for pasting diagrams** In dealing with globular sets, and category-like-structures with cells in arbitrarily high dimensions it will prove useful to have a compact notation for pasting diagrams. To this end we will write globular pasting diagrams without using any arrows, and instead graphically indicate the dimensions and shared boundaries of cells in a table-like presentation. This notation is perhaps easier seen than read, and to that end we give the following table comparing classical cellular renderings to our notation.

<table>
<tr><th>Classical rendering</th><th>Compact notation</th></tr>
<tr><td>
</td><td>$\begin{array}{cc|ccc} \multicolumn{2}{c|}{\alpha} & \multicolumn{3}{c}{\gamma \mid \delta} \\ \hline f & g & h & k & \ell \\ \hline x & & y & & z \end{array}$</td></tr>
<tr><td>
</td><td>$\begin{array}{cc|c} \multicolumn{2}{c|}{\Gamma} & \\ \cline{1-2} \alpha & \beta & \\ \cline{1-2} f & g & h \\ \hline x & y & z \end{array}$</td></tr>
<tr><td>$x \xleftarrow{f} y$ as a 3-dimensional pasting</td><td>$\left\{\left\{ \begin{array}{c} f \\ \hline x \quad y \end{array} \right\}\right\}$</td></tr>
</table>

Now that we have seen some examples of this notation, we point out some of the salient features. First we use curly braces { } to indicate degeneration of an entire pasting diagram, where the number of levels of curly braces indicates the number of times a pasting has been degenerated. Second, looking within a given pasting diagram, the sources and targets of each cell are located to the right and left respectively below the horizontal line. A vertical line indicates that cells have been pasted along a given boundary, where the dimension of the boundary is given by the starting row of the line counting upward from zero and the cell marking that boundary is located beneath the start of the line. Finally, thinking of lines as separating the diagram into regions, multiple cells in a given region are understood to be parallel.

As an added benefit, our notation allows us to render cells in an arbitrary and unknown dimension. For instance, for an $n$-cell $\alpha$ we may draw

$$\begin{array}{cc} \multicolumn{2}{c}{\alpha} \\ \hline f & g \\ \hline \vdots & \vdots \\ \hline x & y \end{array}$$

so as to emphasise its immediate boundary and the eventual 0-cells it lives over.

We conclude this discussion by noting that our notation is, in some sense, a labelled version of the planar dual of Batanin's tree-like notation for unlabelled pasting diagrams [Bat98], recalled around [Lei09, §8.1, diag. (8:7)].

**Two definitions of $\omega$-categories** We reserve the prefix "$\omega$-" in this document for strict-type objects only. As such, we choose the following terminology for our purposes.

**Definition/Notation 2.5.2** (Sense 1). The category $\omega$Cat of $\omega$-categories and $\omega$-functors, is the category of algebras $\widehat{\mathbb{G}}^T$ of the free strict $\omega$-category monad. ◁

However, $\omega$-categories are a limit of an enriching process and so we may "deloop" the above definition find a second form in which it will be convenient to treat $\omega$-categories.

**Definition/Notation 2.5.3** (Sense 2). The category of $\omega$-categories and $\omega$-functors, is the category $\widehat{\mathbb{G}}^T$-Cat, that is, the category of $\widehat{\mathbb{G}}^T$-enriched categories where $\widehat{\mathbb{G}}^T$ is the cartesian monoidal category of algebras for the free strict $\omega$-category monad $T$.

In detail, an **$\omega$-category** $\mathcal{C}$ comprises the data of

(i) a set of objects $\mathrm{ob}\,\mathcal{C}$,

(ii) for each pair of objects $a, b \in \mathrm{ob}\,\mathcal{C}$, an algebra for $T$

$$\big(\mathcal{C}(a,b), h_{a,b} \colon T\mathcal{C}(a,b) \to \mathcal{C}(a,b)\big) \ ,$$

(iii) for each object $a \in \mathrm{ob}\,\mathcal{C}$, a morphism of algebras $\mathrm{id}_a \colon 1 \to \mathcal{C}(a,a)$,

(iv) for each triple of objects $a, b, c \in \mathrm{ob}\,\mathcal{C}$, a morphism of algebras

$$\circ_{a,b,c} \colon \mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c) \ ,$$

such that composition is associative and unital. An $\omega$-**functor** $F\colon \mathcal{C} \to \mathcal{D}$ is thus the data of a function $\operatorname{ob} F\colon \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$ as well, for each pair $a, b \in \operatorname{ob}\mathcal{C}$, a map of algebras $F_{a,b}\colon \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ compatible with identities and composition. ◃

We will take for granted various other aspects of the theory of $\omega$-categories, for instance that reasonable limits and colimits exist, but will endeavour to point these out as we use them. This concludes the background for the present work.

# 3. Proof-relevant categories

The objects of central study in this work, and indeed the motivating example for the framework considered later in Section 5, are the proof-relevant categories.

A proof-relevant category is a higher-category-like structure, with potentially non-trivial cells in arbitrarily high dimensions, and one in which composition is "as weak as can be". By this phrase we mean at first that proof-relevant categories make no fixed assumptions about the nature of composition—there are no prescribed operations or coherences, and composition is applicable to any (globular) pasting of cells. Indeed the very results of composition depend not only on the shape of the pasting of cells as we might expect, but also on the cells themselves. There may be fewer or more ways to compose some given cells in a fixed arrangement than other cells in the same arrangement. In the zero-ary case of composition, we see therefore that there are no fixed identity cells. Instead a proof-relevant category houses potentially many distinct identity cells on a given cell, all of which behave appropriately.

This is not to imply that a proof-relevant category is "wild" and indeed despite composition being a multi-valued relation which is highly sensitive to the particulars of its input, the results are always coherent. The reason this is so is because proof-relevant categories harbour a calculus of proofs. In a departure from other models of higher-category-like structures, proof-relevant categories have a notion of what it means to prove that a certain cell is an evaluation of a pasting of other cells *without* requiring these proofs to be of the same "sort" as the cells upon which they are predicated. That is, a proof-relevant category has two fundamentally different aspects: cells which may be arranged into pastings, and proofs which reason about arrangements of cells and determine evaluations.

These proofs may be concatenated in a certain fashion to derive compound proofs, those proofs that witness that an evaluation of a pasting, itself formed of evaluations, is an evaluation of the overall pasting. There are also reflexivity proofs which prove that every cell evaluates to itself. Using this proof calculus and an appropriate notion of equivalence, it is possible to prove several direct formulations of composition coherence. That is, despite there being many possible ways to evaluate a fixed pasting, all of the results are appropriately equivalent.

With this in mind, this section is structured as follows. In Section 3.1 we state the definition of a proof-relevant category and guide the reader through understanding its basic features. In Section 3.2 we attend to the notions of identities and equivalences, and prove some basic interactions between these and the ambient structure of a proof-relevant category. Finally in Section 3.3 we give several classes of examples of proof-relevant categories.

*Remark.* Readers familiar with the content of [Lei09] might recognise our Definition 3.1.1 for a proof-relevant category below as a minor variation of Leinster's model $L'$ first appearing in [Lei01, Def. $L'$], and later in [Lei09, §10.2] under the name contracitble multicategory. The are three differences between our definition and that of Leinster. The first is that in $L'$ the domain leg is required to be bijective on objects. By contrast, we have no such requirement—see Remark 3.2.17 for a related discussion. The second is that $L'$ relies on $\mathsf{refl}$ to give, by a similar argument to Lemma 2.4.3, at least one way to evaluate every pasting. By contrast we demand a split contraction instead of a contraction and so Lemma 2.4.3 applies verbatim—this introduces some amount of "redundancy" and we attend to that in Remark 3.2.21. Finally and most importantly, distinguishing our presentation from the prior work is the presence of a more general framework for generating this and other related definitions. This framework is the subject of Section 5.

Once we have seen the machinery of that section we will be able to realise the below definition in new terms. In the language to come, a proof-relevant category is a proof-relevant algebra for the terminal $T$-category $1 \leftarrow T1 = T1$. This phrasing is intended to be contrasted with the analogous version for strict $\omega$-categories, viz., a strict $\omega$-category is a strict algebra for the terminal $T$-category $1 \leftarrow T1 = T1$. By varying the base $T$-category as in Section 5.4, we may extract a definition of proof-relevant functor (Section 4) and higher transfors, as well as algebraic notions such as compositions of these transfors, proof-relevant monads, and so on. ◃

## 3.1. The definition

Let $T$ be the free strict $\omega$-category monad of Section 2.5 on the category $\widehat{\mathbb{G}}$ of globular sets. In this context, we make the following definition.

**Definition 3.1.1.** A **proof-relevant category** is the data of a $T$-category $\mathbb{A}$ whose domain leg $d\colon P \to TA$ is equipped with the data of a split contraction. ◃

In what follows we will choose to denote proof-relevant categories by blackboard bold letters near the start of the alphabet: $\mathbb{A}$, $\mathbb{B}$, and so on.

*Remark* 3.1.2. It may prove useful to explicate here the structure of a proof-relevant category by unwinding Definition 2.2.5. A proof-relevant category $\mathbb{A}$ comprises globular sets $A, P$, globular set morphisms $c\colon P \to A$ and $d\colon P \to TA$, as well as the structure of a split contraction (Definition 2.4.1) on $d$. In addition there are morphisms of globular sets $\mathsf{refl}\colon A \to P$ and $\mathsf{comp}\colon P \circ P \to P$ rendering commutative the following diagrams, and furthermore satisfying associativity and unitality laws.

$$\begin{tikzcd} A & P \arrow[l, "c"'] \arrow[r, "d"] & TA \\ & A \arrow[ul, equal] \arrow[u, "\mathsf{refl}"] \arrow[ur, "\eta_A"'] & \end{tikzcd} \qquad \begin{tikzcd} A & P \arrow[l, "c"'] \arrow[r, "d"] & TA \\ & P \circ P \arrow[ul, "cp"] \arrow[u, "\mathsf{comp}"] \arrow[ur, "\mu_A(Td)q"'] & \end{tikzcd}$$

In order to make more efficient use of space, when multiple or small instances of $\mathsf{comp}$ are to be rendered we shall draw the map $\mathsf{comp}$ instead as $\square$. ◃

*Notation* 3.1.3. Let us introduce some terminology to make the distinction between the globular sets $A$ and $P$ more clear. When we refer to a **cell** of a proof-relevant category, we always mean an $n$-cell of $A$. When we refer to a **proof**, we always mean an $n$-cell of $P$. ◁

While we may well have *defined* the things we wish to call proof-relevant categories we have yet to state what they *are*, or perhaps more accurately, how we wish to understand them. Such an explanation will be the subject of our immediate attention, after which we will turn our attention to various examples and, in later chapters, aspects of their theory.

### 3.1.1 Understanding the model

Let us begin by observing that we have a globular morphism $\langle c, d\rangle \colon P \to A \times TA$ and so we are permitted to look at the fibres over pairs $(k,\pi) \in (A \times TA)_n$. That is, for an $n$-pasting diagram $\pi$ and an $n$-cell $k$ we may examine the $n$-cells $p \in P$ above $(k,\pi)$ and we shall consider these cells to be *proofs $p$ that the specified $n$-pasting $\pi$ evaluates to the specified $n$-cell $k$.*

For example, if $\pi$ is the 1-pasting below-right and $k$ is the 1-cell below-left, then a 1-cell $p \in P$ above $(k,\pi)$ should be understood as *a* proof that $k$ is *an* evaluation of the pasting of $f, g$, and $h$.

$$\begin{tikzcd}[column sep=small] & p \in P_1 \arrow[dl, "c"'] \arrow[dr, "d"] & & \pi \\ \boxed{w \xleftarrow{k} z} \in A_1 & & TA_1 \ni \boxed{w \xleftarrow{f} x \xleftarrow{g} y \xleftarrow{h} z} \end{tikzcd}$$

We will choose to write $p \in P_1$ as an annotated implication arrow from $\pi$ to $k$,

$$w \xleftarrow{k} z \quad \overset{\frac{p}{q \;\; r}}{\Longleftarrow} \quad w \xleftarrow{f} x \xleftarrow{g} y \xleftarrow{h} z$$

so as to reinforce the understanding that $p$ is a proof. There may in general be many different 1-cells $p \in P$ over the pair $(k,\pi)$, intuitively corresponding to the various ways of composing $f$, $g$, and $h$ specifically—that is, the set of proofs of evaluation varies not only with the shape of the pasting $\pi$, but also with the cells themselves involved in the pasting and the intended evaluation, $k$.

There is more yet, for $P$ is a globular set. Let us consider the situation in the below schematic illustration.

$$\begin{tikzcd}[column sep=small] & \rho \in P_2 \arrow[dl, "c"'] \arrow[dr, "d"] & & \Pi \\ \boxed{\begin{tikzcd} w & z \arrow[l, bend right, "k"'] \arrow[l, bend left, "k'"] \arrow[l, phantom, "\beta\Uparrow"] \end{tikzcd}} \in A_2 & & TA_2 \ni \boxed{\begin{tikzcd} w & x \arrow[l, "f"'] & y \arrow[l, "g"'] & z \arrow[l, bend right, "h"'] \arrow[l, bend left, "h'"] \arrow[l, phantom, "\alpha\Uparrow"] \end{tikzcd}} \end{tikzcd}$$

As $\rho \in P_2$ we know that it must have source and target 1-cells of $P_1$, let us call these $p :\equiv \mathsf{target}\,\rho$ and $p' :\equiv \mathsf{source}\,\rho$. On our account then, $\rho$ is a proof with *source and target proofs.* More still, as $d$ and $c$ are globular morphisms we see that, for instance, $dp = d\,\mathsf{target}\,\rho = \mathsf{target}\,d\rho = \mathsf{target}\,\Pi$ where $\mathsf{target}\,\Pi = \pi$ by inspection. That is, the data of a proof $\rho \in P_2$ above the pair $(\beta, \Pi)$ includes also *proofs that the boundaries of* $\Pi$ *evaluate to the boundaries of* $\beta$. In this case the intuition is that to give a proof that $\beta$ is an evaluation of the whiskering of the 2-cell $\alpha$ with $f$ and $g$ we must first settle how we intend to compose the boundaries.

We wish to emphasise that proof-relevant categories may be somewhat different in formulation from other models of higher categories in that, as we have just seen, the available ways to evaluate a pasting *depend on the cells involved in the pasting.* This is not usually so, and the reader is perhaps used to models where there are either specified operations which depend only on the *shape* of the pasting of cells or models that do not make this dependence clear—those which demand only that *an* evaluation must exist.

Finally we must admit a slight, intentional error of presentation: we have tacitly made use of the convenient simplification that $c(\mathsf{source}\, p) = c(r) = z = d(q) = d(\mathsf{source}\, p)$ and so on, that is, the 0-cell boundaries of $\pi$ and $k$ are the same. Of course there is no reason in general that this should hold true, and we will address the nature of these object proofs in Remark 3.2.17.

### 3.1.2 A tour of the features

With a basic understanding established we turn now to explore the idea that composition is determined only up to equivalence in a proof-relevant category. Rather than treating this concept in strict generality we will, for the time being, satisfy ourselves by constraining our attention to the particular case of binary composition of 1-cells, and in so doing cement our intuition for Section 3.2 which rigorously deals with these notions. Although binary composition of 1-cells is certainly a special case, it is nevertheless representative and makes use of all the features of a proof-relevant category: globularity, $T$, the split contraction structure on $d$, and the operations $\mathsf{refl}$ and $\mathsf{comp}$.

Suppose then that $\mathbb{A}$ is a proof-relevant category and that we have 0-cells $x, y, z \in A_0$ and 1-cells $f, g \in A_1$ with sources and targets as indicated below, forming the 1-pasting $\pi \in TA_1$.

$$\pi :\equiv \boxed{x \xleftarrow{f} y \xleftarrow{g} z}$$

Now let us attend to the subtlety mentioned at the end of the previous section. Instead of predicating our discussion on any proof of an evaluation of the pasting $\pi$, that is, any element of the pre-image $d^{\leftarrow}\{\pi\}$, we will instead constrain our attention to those which evaluate $\pi$ to a 1-cell with the same 0-cell boundary—the expected result of traditional binary composition. In order to do this we must appeal to the morphism $\mathsf{refl}\colon A \to P$.

Recall that $TA_0 \equiv A_0$ and $(\eta_A)_0 \equiv \mathrm{id}_{A_0}$ so that $\mathsf{refl}$ is a section of both $d$ and $c$ on 0-cells. For brevity and where it will cause no confusion, we will choose to name the proofs $\mathsf{refl}_\alpha \in P$ by the cells $\alpha \in A$ to which $\mathsf{refl}$ has been applied. With this convention, a pictorial rendition of the reflexivity proof for a 0-cell $x \in A_0$ is therefore the following.

$$x \overset{x}{\Longleftarrow} x$$

In order to emulate the expected nature of binary composition we will thus predicate our discussion on those $p$ proofs appearing in the set $p \in P_1$ whose boundaries are all precisely $\partial p :\equiv \langle x, z \rangle \colon \partial_1 \to P$. By design such proofs will prove that $\pi$ evaluates to 1-cells $z \leftarrow x$.

While any proof $p \in P_1$ with boundary $\partial p$ would give a satisfactory account of binary composition, we have not yet assured ourselves that there exist any such. To see that there must be, we may appeal to the split contraction structure on $d \colon P \to TA$. Observe that in fixing the pasting $\pi \colon 2_1 \to TA$ and the boundary $\partial p \colon \partial_1 \to P$ we have in fact given the data of a lifting problem

$$\begin{tikzcd}
\partial_1 \arrow[r, "\partial p"] \arrow[d, mapsto] & P \arrow[d, "d"] \\
2_1 \arrow[r, "\pi"'] & TA
\end{tikzcd},$$

and so the split contraction data on $d$ specifies a lift $\kappa(\pi, \partial p) \colon 2_1 \to P$ in the above square. That is, we are assured of the existence of *at least one* proof $\kappa(\pi, \partial p) \in P_1$ which proves that $\pi$ evaluates to a 1-cell $c(\kappa(\pi, \partial p))$ as below.

$$x \xleftarrow{c(\kappa)} z \quad \overset{\frac{\kappa}{x\ z}}{\Longleftarrow} \quad x \xleftarrow{f} y \xleftarrow{g} z$$

Now that we have demonstrated that there is always at least one proof allowing us to form a traditional binary composite of two 1-cells in a proof-relevant category, we may turn our attention to the possibility of the existence other such proofs. That is, suppose that we had 1-cells $h, h' \in A_1$ from $z$ to $x$, as well as proofs $p, p' \in P_1$ both of whose boundary is $\langle x, z \rangle$ and living over the pairs $(h, \pi)$ and $(h', \pi)$ respectively. That is, suppose that we had proofs $p, p'$ that $\pi$ evaluates to $h$ and $h'$ respectively.

$$x \xleftarrow{h} z \quad \overset{\frac{p}{x\ z}}{\Longleftarrow} \quad x \xleftarrow{f} y \xleftarrow{g} z \qquad \text{and} \qquad x \xleftarrow{h'} z \quad \overset{\frac{p'}{x\ z}}{\Longleftarrow} \quad x \xleftarrow{f} y \xleftarrow{g} z$$

Were this proof-relevant category to have “single-valued composition” or “composition operations”, we would expect to be able to deduce that $h = h'$. In more general cases, we might expect instead only that $h$ and $h'$ be suitably “equivalent”—a term whose precise meaning we shall soon explore.

Proof-relevant categories as we have defined them allow us to deduce precisely such an equivalence. To see this, consider the pasting $\pi$ above not as a 1-cell $\pi \colon 2_1 \to TA$ but as a degenerate 2-dimensional pasting $\{\pi\} \colon 2_2 \to TA$. Observe then that we may construct the lifting problem

$$\begin{tikzcd} \partial_2 \arrow[r, "{\langle p,p' \rangle}"] \arrow[d, rightarrowtail] & P \arrow[d, "d"] \\ 2_2 \arrow[r, "{\{\pi\}}"'] & TA \end{tikzcd} ,$$

and we are afforded a solution $\kappa\big(\{\pi\}, \langle p,p' \rangle\big) \colon 2_2 \to P$ by the structure of the split contraction on $d$. That is, we are assured of at least one proof of the form

$$\begin{tikzcd} x \arrow[r, phantom, "\beta\Uparrow"] & z \arrow[l, bend right, "h"'] \arrow[l, bend left, "h'"] \end{tikzcd} \quad \overset{\dfrac{\kappa}{\dfrac{p \;\; p'}{x \;\; z}}}{\Longleftarrow} \quad \left\{ x \xleftarrow{f} y \xleftarrow{g} z \right\} .$$

Similar arguments, but with the roles of $h$ and $h'$ interchanged, give the existence of an opposed 2-cell $\beta'$ from $h$ to $h'$. To show that these cells are the start the data of an equivalence between $h$ and $h'$ we need to compare *their* composites to identity cells.

Herein we encounter a perhaps novel subtlety of proof-relevant categories. In some contemporary models of higher categories, despite ordinary composition being appropriately weak, special treatment is given to identity cells: they are *specified* cells. However, identity cells are the zero-ary case of composition and so we suggest that it is odd that there are uniquely chosen results for these—even if the manner in which they behave as identities is somehow weak. Proof-relevant categories, as a result of being fully weak and unbiased, do *not* come equipped with a privileged set of identity cells inasmuch as the calculus of compositions and proofs is concerned. As we shall soon see, while it is true that there is an assignment of cells to “endo-equivalences” as a consequence of the constructive existence of composites, there may be many cells which behave just like these and no particular choice is privileged in any further way. A rigorous expansion on this idea is given in Section 3.2 and specifically Definition 3.2.2. In this way, we might say that proof-relevant categories do not even have a notion of identity cell as distinct from other, comparable equivalences.

To treat rigorously the notions of identities and equivalences we will need to appeal to the last feature of proof-relevant categories on our tour: proof composition. Let us therefore temporarily shelve the discussion about uniqueness of evaluations and attend to this.

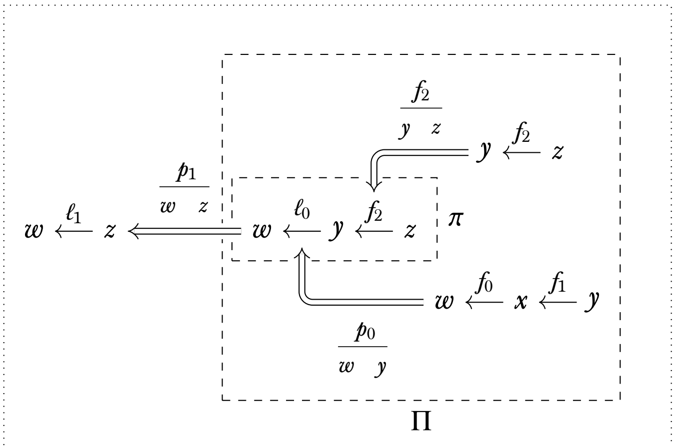
In our discussion so far of binary composition, we have not yet considered the question of iteration and whether some form of associativity might hold. That is, suppose that we wish to compose three 1-cells, $f_0$, $f_1$, and $f_2$. Should we wish to do this binarily, there are two possible associations which we now schematically indicate.

$$
\begin{array}{|c|}
\hline
\textbf{Binary compositions of } w \xleftarrow{f_0} x \xleftarrow{f_1} y \xleftarrow{f_2} z \\
\begin{array}{c|c}
\text{Left associated} & \text{Right associated} \\
\hline
w \xleftarrow{\ell_0} y \quad \overset{\frac{p_0}{w\ \ y}}{\Longleftarrow} \quad w \xleftarrow{f_0} x \xleftarrow{f_1} y & x \xleftarrow{r_0} z \quad \overset{\frac{q_0}{x\ \ z}}{\Longleftarrow} \quad x \xleftarrow{f_1} y \xleftarrow{f_2} z \\
x \xleftarrow{\ell_1} z \quad \overset{\frac{p_1}{x\ \ z}}{\Longleftarrow} \quad w \xleftarrow{\ell_0} y \xleftarrow{f_2} z & x \xleftarrow{r_1} z \quad \overset{\frac{q_1}{w\ \ z}}{\Longleftarrow} \quad w \xleftarrow{f_0} x \xleftarrow{r_0} z
\end{array} \\
\hline
\end{array}
\tag{3.1.4}
$$

Doubtlessly our proof-relevant category should recognise both $\ell_1$ and $r_1$ as evaluations of the pasting $\pi :\equiv (f_0|f_1|f_2)$, but none of the proofs we have above have $\pi$ as their domain and therefore none of these proofs directly prove this fact. Nevertheless, there are proofs—derivable from the information above—which do prove that $\ell_1$ and $r_1$ are evaluations of the pasting $\pi$. To see this, let us first recall the definition of $P \circ P$ as the below span between $A$ and $TA$ in globular sets, whose apex is the vertex of the indicated pullback.

$$
\begin{tikzcd}
& P \circ P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TP \arrow[r, "Td"] \arrow[d, "Tc"] & T^2A \arrow[r, "\mu_A"] & TA \\
A & P \arrow[l, "c"] \arrow[r, "d"'] & TA & &
\end{tikzcd}
\tag{3.1.5}
$$

Thus an $n$-cell in $P \circ P$ is a pair of a proof $\alpha \overset{p}{\Leftarrow} \pi$ as well as a pasting of proofs $\Pi \in TP$ such that the pasting of the evaluation cells $(Tc)\Pi$ is $\pi$. For example and using the notation of (3.1.4) above, we may schematically illustrate the following 1-cell of $P \circ P$.



$\in (P \circ P)_1$

In the above we see that the 1-cell named in $TP$ is in fact

$$
\Pi :\equiv \boxed{\frac{p_0 \mid f_2}{w \quad y \quad z}},
$$

but also that we have the equality $\pi = (Tc)\Pi$. Thus the pair $(p_1, \Pi)$ constitutes a 1-cell of $P \circ P$. With that in hand, proof composition (Remark 3.1.2) is an operation

$\mathsf{comp}\colon P \circ P \to P$ and so we may apply it to the pair $(p_1, \Pi)$ to obtain a new proof $p_1 \square \Pi \in P$.

The axioms of a proof-relevant category assure us that we may compute $c\,\mathsf{comp}$ and $d\,\mathsf{comp}$ equally by following the legs from the vertex in (3.1.5) to $A$ and $TA$ respectively. In our example therefore we see that $p_1 \square \Pi$ is in fact a proof that $\ell_1$ is an evaluation of the triple pasting,

$$w \xleftarrow{\ell_1} z \quad \overset{\frac{p_1 \,\square\, \Pi}{w \qquad z}}{\Longleftarrow\!=\!=\!=} \quad w \xleftarrow{f_0} x \xleftarrow{f_1} y \xleftarrow{f_2} z \ .$$

Implicit in our writing the proof $p_1 \square \Pi$ as a 1-cell above we have made use of the unitality of proof composition: $\mathsf{refl}_\alpha \square p = p$ and $p \square (\mathsf{refl}) = p$ where in the latter we take $(\mathsf{refl})$ to stand for a pasting made entirely of reflexivity proofs.

To conclude this section, note that we could have also used the right-associated data of 3.1.4 to derive a proof that $r_1$ is an evaluation of the triple pasting

$$w \xleftarrow{r_1} z \quad \overset{\frac{q_1 \,\square\, \Pi'}{w \qquad z}}{\Longleftarrow\!=\!=\!=} \quad w \xleftarrow{f_0} x \xleftarrow{f_1} y \xleftarrow{f_2} z \ ,$$

where $\Pi' :\equiv \frac{f_0 \;\mid\; r_0}{w \quad x \quad z}$. We therefore once again find ourselves in the position of having two proofs that two cells are the evaluation of a common pasting diagram. However, we have now seen enough of the elementary details of proof-relevant categories that we are equipped to rigorously define a notion of equivalence and to prove that all such cells are equivalent.

## 3.2. Identities and equivalences

The discussion of the previous section has highlighted that any notion of equivalence in a proof-relevant category depends on a logically prior notion of identity. This section then is dedicated to the definition of identities, equivalences, and some of their early interactions with the rest of the structure of a proof-relevant category.

As we will now be defining and proving statements in proof-relevant categories, let us fix some suggestive notation for proofs and evaluations.

*Notation* 3.2.1. When given an explicit pasting of cells, say $\pi :\equiv (x \xleftarrow{f} y \xleftarrow{g} z)$, we shall sometimes find it convenient to suppose that a proof evaluating $\pi$ names a cell with a suggestive name, as in

$$x \xleftarrow{fg} z \quad \overset{\frac{p}{x \; z}}{\Longleftarrow\!=} \quad x \xleftarrow{f} y \xleftarrow{g} z \ .$$

By this notation we do not mean to imply that there are composition operations, rather our use of $fg$ is simply a more memorable name for what should rightfully be called $c(p)$—the cell which $p$ proves to be an evaluation of $\pi$. ◃

**Definition 3.2.2.** Let $\mathbb{A}$ be a proof-relevant category, and $\alpha$ be an $n$-cell. An **identity cell on** $\alpha$ is the data of a proof

$$1_\alpha \xLeftarrow[\alpha\ \alpha]{p} \{\alpha\}\ ,$$

such that $\mathsf{source}\, p = \mathsf{refl}_\alpha = \mathsf{target}\, p$ and $dp = \{\alpha\}$, where $\{\alpha\}$ is the degenerate $(n+1)$-cell on $\alpha$. ◃

**Lemma 3.2.3.** *In a proof-relevant category every cell has at least one identity cell.*

*Proof.* From each $n$-cell $\alpha$ we may form the boundary $\langle\alpha,\alpha\rangle\colon 2_{n+1} \to P$ using $\mathsf{refl}$, whose image under $d$ in $TA$ is the boundary of the $(n+1)$-cell $\{\alpha\}\colon 2_{n+1} \to TA$. Thus we obtain the lifting problem

$$\begin{tikzcd} \partial_n \arrow[r, "\langle\alpha,\alpha\rangle"] \arrow[d, tail] & P \arrow[d, "d"] \\ 2_n \arrow[r, "\{\alpha\}"'] & TA \end{tikzcd}\ ,$$

for which the split contraction $\kappa$ on $d$ specifies a solution $2_{n+1} \to P$ and therefore an identity cell. ■

**Definition 3.2.4.** Let $x$ and $y$ be two cells in a proof-relevant category of the same dimension. An **equivalence** $x \sim y$ **between** $x$ **and** $y$ is defined to comprise the following data:

(i) a pair of opposed cells $x \xleftarrow{\alpha} y$ and $y \xleftarrow{\beta} x$,

(ii) a pair of evaluations

$$x \xleftarrow{\alpha\circ\beta} x \quad \xLeftarrow[x\ x]{p} \quad x \xleftarrow{\alpha} y \xleftarrow{\beta} x \qquad \text{and} \qquad y \xleftarrow{\beta\circ\alpha} y \quad \xLeftarrow[y\ y]{q} \quad y \xleftarrow{\beta} x \xleftarrow{\alpha} y$$

where the sources and targets of $p$ and $q$ are required to be $\mathsf{refl}_x$ and $\mathsf{refl}_y$ as indicated,

(iii) a pair of identity cells

$$1_x \xLeftarrow[x\ x]{a} \{x\} \qquad \text{and} \qquad 1_y \xLeftarrow[y\ y]{b} \{y\}\ , \text{and}$$

(iv) two further equivalences, $1_x \sim \alpha\circ\beta$ and $\beta\circ\alpha \sim 1_y$.

We term the cells $\alpha$ and $\beta$ the **mediating cells** and the cells $1_x$ and $1_y$ the **reference identities**. ◃

*Remark* 3.2.5. For the reader who eschews coinductive definitions, we note that it is possible to state the definition of equivalence inductively instead. To do this we would define an $m$-truncated equivalence, as well as a truncation function from $(m+1)$-truncated equivalences to $m$-truncated equivalences, and then declare equivalences to be the limit over this $\omega^{\mathrm{op}}$ diagram. That is, an equivalence is that which is an $m$-truncated equivalence for all $m \in \mathbb{N}$, compatibly so with taking the $(m+1)$-truncation of its data to the $m$-truncation of its data. ◃

*Remark* 3.2.6. Although the definition of equivalence we gave above is fairly loose in the structure it requires of its data, it is nevertheless not entirely free of constraints. That is, we might imagine that there happen to be pairs of opposed cells in nature which behave in some sense like the mediating cells of equivalences, but the higher data are for instance predicated on evaluations that do not meet the conditions in Definition 3.2.4 (ii). As we shall suggest in Conjecture 3.2.19, a class of these sorts of equivalences ought to be equivalent to ours, so that we have not lost any generality.

On the other hand, the sceptical reader may be uncomfortable with the idea that our equivalences are as loose as they are: we are permitted to use any and all reference identities at any and all stages of the data. Surely, such a reader might argue, it would be best if we more tightly constrained the data to avoid “wild” equivalences. As it happens, in Lemma 3.2.18 we shall see that such a notion of equivalence is in fact equivalent to ours, so we have not been overly general in our specification. ◃

We also wish to point out that this definition is compatible with the data of every proof-relevant category in the following sense. As we shall soon see in Lemma 3.3.1, by their very nature proof-relevant categories are required to have cells and proofs in every dimension if they have any cells at all—in a sense, iterated identities—and so there is no way to exhaust the available data and “run out” before an entire equivalence may be given.

Let us now turn to three of the foundational facets of equivalences: equivalences witness all composition coherences, identities generate equivalences, and equivalences are suitably reflexive, symmetric, and transitive.

To begin, let us prove that this notion of equivalence suitably captures at least a two senses of composition coherence in a proof-relevant category. First, any two parallel proofs that a fixed pasting may be evaluated in any two corresponding parallel ways yield equivalent evaluations. We invite the reader to compare the truncated, corresponding result for bicategories (or monoidal categories): any pair of choices for composing a pasting of 1-cells (resp. pair of choices of tensor applications) gives rise to an isomorphism between their results.

**Lemma 3.2.7.** *Let $\pi$ be a pasting of cells in a proof-relevant category. If there are two parallel proofs*

$$\alpha \overset{p}{\Leftarrow} \pi \qquad \text{and} \qquad \beta \overset{q}{\Leftarrow} \pi\,,$$

*then there is the data of an equivalence $\alpha \sim \beta$ on the two evaluations of $\pi$.*

*Remark* 3.2.8. There is an evident symmetry in the hypothesis of Lemma 3.2.7 given by interchanging the roles of $\alpha$ and $\beta$. Thus we are entitled to conclude additionally the existence of equivalence data $\beta \sim \alpha$. As we shall see later in Lemma 3.2.12 (ii) this is of course part of a more general fact: equivalences are appropriately symmetric notions. ◁

*Proof* (Lemma 3.2.7). As demanded by our coinductive definition of equivalence, we will give a coinductive proof of this statement. That is, we shall construct explicitly the data of Definitions 3.2.4 (ii) and 3.2.4 (iii) and then use the process we have just outlined on a new set of cells to give the data of Definition 3.2.4 (iv).

To that end, because $p$ and $q$ are parallel let us fix the evaluations

$$\alpha \overset{\Theta}{\leftarrow} \beta \quad \overset{\frac{\kappa(p,q)}{p \qquad q}}{\Longleftarrow} \quad \{\pi\} \qquad \text{and} \qquad \beta \overset{\Phi}{\leftarrow} \alpha \quad \overset{\frac{\kappa(q,p)}{q \qquad p}}{\Longleftarrow} \quad \{\pi\}$$

given by the split contraction $\kappa$. These will serve as our mediating cells and so let us determine also evaluations

$$\alpha \overset{\Theta\circ\Phi}{\longleftarrow} \alpha \quad \overset{\frac{\kappa(\alpha)}{\alpha \qquad \alpha}}{\Longleftarrow} \quad \alpha \overset{\Theta}{\leftarrow} \beta \overset{\Phi}{\leftarrow} \alpha \qquad \text{and} \qquad \beta \overset{\Phi\circ\Theta}{\longleftarrow} \beta \quad \overset{\frac{\kappa(\beta)}{\beta \qquad \beta}}{\Longleftarrow} \quad \beta \overset{\Phi}{\leftarrow} \alpha \overset{\Theta}{\leftarrow} \beta$$

from the split contraction $\kappa$. Observe that $\kappa(p,q)$ and $\kappa(q,p)$ are paste-able proofs, and liable for the action of proof composition with $\kappa(\alpha)$.

$$\begin{tikzcd}
 & & & & \{\pi\} \arrow[dl, Rightarrow, bend right, "\frac{\kappa(p,q)}{p \qquad q}"'] \\
\alpha & \alpha \arrow[l, "\Theta \circ \Phi"'] & \alpha \arrow[l, Rightarrow, "\frac{\kappa(\alpha)}{\alpha \qquad \alpha}"'] & \beta \arrow[l, "\Theta"'] & \alpha \arrow[l, "\Phi"'] \\
 & & & & \{\pi\} \arrow[ul, Rightarrow, bend left, "\frac{\kappa(q,p)}{q \qquad p}"]
\end{tikzcd}$$

A similar property is true of $\kappa(\beta)$, with a pasting of $\kappa(q,p)$ and $\kappa(p,q)$. Thus, by proof composition we may obtain the following proofs that the evaluations $\Theta \circ \Phi$ and $\Phi \circ \Theta$ are both evaluations of $\{\pi\}$.

$$\alpha \overset{\Theta\circ\Phi}{\longleftarrow} \alpha \quad \overset{\frac{\kappa(\alpha)\,\square\,(\kappa(p,q)|\kappa(q,p))}{p \qquad\qquad\qquad\qquad p}}{\Longleftarrow\!=\!=\!=\!=} \quad \{\pi\} \quad \text{and} \quad \beta \overset{\Phi\circ\Theta}{\longleftarrow} \beta \quad \overset{\frac{\kappa(\beta)\,\square\,(\kappa(q,p)|\kappa(p,q))}{q \qquad\qquad\qquad\qquad q}}{\Longleftarrow\!=\!=\!=\!=} \quad \{\pi\}$$

These data fulfil the burdens of Definition 3.2.4 (ii). Now let us appeal to Lemma 3.2.3 to obtain identities

$$\alpha \overset{1_\alpha}{\longleftarrow} \alpha \quad \overset{\frac{r}{\alpha\ \alpha}}{\Longleftarrow} \quad \{\alpha\} \qquad \text{and} \qquad \beta \overset{1_\beta}{\longleftarrow} \beta \quad \overset{\frac{s}{\beta\ \beta}}{\Longleftarrow} \quad \{\beta\}$$

which fulfils the requirement of Definition 3.2.4 (iii). To give equivalences demanded by Definition 3.2.4 (iv), viz., the data of $1_\alpha \sim \Theta \circ \Phi$ and $\Phi \circ \Theta \sim 1_\beta$, we wish to make a coinductive appeal to the proof of this lemma thus far. However, in order to do that we need to realise $1_\alpha$ and $1_\beta$ as evaluations of $\{\pi\}$ so that they may be compared to $\Theta \circ \Phi$ and $\Phi \circ \Theta$ respectively.

This is possible for $r$ and $\{p\}$, and $s$ and $\{q\}$ are paste-able proofs, and therefore we may construct the following proof composites

$$\alpha \xleftarrow{1_\alpha} \alpha \quad \overset{\overline{r \,\square\, \{p\}}}{\underset{}{\overset{p \qquad p}{\Longleftarrow\!\!=\!\!=}}} \quad \{\pi\} \qquad \text{and} \qquad \beta \xleftarrow{1_\beta} \beta \quad \overset{\overline{s \,\square\, \{q\}}}{\underset{}{\overset{q \qquad q}{\Longleftarrow\!\!=\!\!=}}} \quad \{\pi\}\ ,$$

thereby completing the proof. ■

For the second aspect of composition coherence, let us determine that evaluating pastings which contain identities generates equivalent values to evaluating pastings without the identities inserted.

**Lemma 3.2.9.** *In a proof-relevant category identities enjoy a form of indistinguishability for composition. That is, suppose that $\pi$ is a pasting of cells in a proof-relevant category in which identities appear as top-dimensional cells. Let us write $\widehat{\pi}$ for the pasting derived from $\pi$ by omitting all of the top-dimensional identities and replacing them with degenerate versions of their sources (or equally targets). Then for all evaluations*

$$\alpha \overset{q}{\Leftarrow} \pi \qquad \textit{and} \qquad \beta \overset{r}{\Leftarrow} \widehat{\pi}$$

*such that $q$ and $r$ are parallel, there is an equivalence $\alpha \sim \beta$.*

As an example of the procedure of this lemma, let us specialise the claim to a simple 1-pasting.

**Example 3.2.10**

Suppose that the are cells $x \overset{f}{\leftarrow} y \overset{g}{\leftarrow} z$. For all identities $1_y \overset{p}{\Leftarrow} \{y\}$ and evaluations

$$x \xleftarrow{f \circ 1_y \circ g} z \quad \overset{\overline{q}}{\underset{}{\overset{x\ z}{\Longleftarrow}}} \quad x \overset{f}{\leftarrow} y \overset{1_y}{\longleftarrow} y \overset{g}{\leftarrow} z \qquad \text{and} \qquad x \xleftarrow{f \circ g} z \quad \overset{\overline{r}}{\underset{}{\overset{x\ z}{\Longleftarrow}}} \quad x \overset{f}{\leftarrow} y \overset{g}{\leftarrow} z\ ,$$

there is the data of an equivalence $(f \circ 1_y \circ g) \sim f \circ g$.

*Proof* (Lemma 3.2.9). Let us write $p_i$ variously for the proofs that the identities in $\pi$ are evaluations of degenerate pastings of their sources (or equally targets). Note that there is a proof composition $q \,\square\, (p_i)$, where $(p_i)$ is the pasting obtained from $\pi$ by replacing each non-identity cell with reflexevity and each identity cell with its proof $p_i$. The proof thus obtained is parallel to $r$ as proofs evaluating $\widehat{\pi}$, thus by Lemma 3.2.7 the evaluations are equivalent. ■

Next in the important interactions between equivalences, identities, and composition is the fact that all identities mediate equivalences.

**Lemma 3.2.11.** *For any cell $x$ and any identity $1_x \overset{p}{\Leftarrow} \{x\}$, there is an extension of $1_x$ to the mediating cells of an equivalence $x \sim x$.*

*Proof.* We have already determined that our mediating cells should both be $1_x$, and so we must therefore fix evaluations, reference identities, and higher equivalences thereupon. The split contraction $\kappa$ guarantees an evaluation

$$x \xleftarrow{1_x \circ 1_x} x \overset{\frac{\kappa}{x\ x}}{\Longleftarrow} x \xleftarrow{1_x} x \xleftarrow{1_x} x \ ,$$

which we take for both evaluations of $1_x$ pasted with itself. For reference identities we select of course the given data of $1_x$. Now to complete the construction we must give equivalences $1_x \sim 1_x \circ 1_x$ and $1_x \circ 1_x \sim 1_x$. Note however that proof composition witnesses that $1_x \circ 1_x$ is an evaluation of $\{x\}$

$$x \xleftarrow{1_x \circ 1_x} x \overset{\frac{\kappa \,\square\, (p|p)}{x \qquad x}}{\Longleftarrow} \{x\} \xleftarrow{\ \mathsf{comp}\ } \left\{ \begin{tikzcd} & & & & \{x\} \arrow[dl, Rightarrow, "\frac{p}{x\ x}"'] \\ x & x \arrow[l, "1_x \circ 1_x"'] & x \arrow[l, Leftarrow, "\frac{\kappa}{x\ x}"'] & x \arrow[l, "1_x"'] & x \arrow[l, "1_x"'] \\ & & & & \{x\} \arrow[ul, Rightarrow, "\frac{p}{x\ x}"] \end{tikzcd} \right.$$
,

so that Lemma 3.2.7 and Remark 3.2.8 applied to the two evaluations $1_x$ and $1_x \circ 1_x$ of $\{x\}$ give the desired equivalence data. ∎

The third important property of equivalences is that they satisfy the expected implications of an equivalence relation. Two of these three facts are straightforward, but transitivity is a more delicate matter and so we postpone it to the next lemma.

**Lemma 3.2.12.** *Let $x$, $y$, and $z$ be cells of the same dimension in a proof-relevant category. Then,*

*(i) there is a equivalence $x \sim x$,*

*(ii) if there is an equivalence $x \sim y$ then there is an equivalence $y \sim x$, and*

*Proof* (Lemma 3.2.12). The first claim is Lemma 3.2.11 applied to the identities of Lemma 3.2.3.

Next suppose that we have the data of an equivalence $x \sim y$. We shall coinductively construct the data of an equivalence $y \sim x$. First, let us interchange the roles of the mediating cells $\alpha$ and $\beta$ of $x \sim y$ so as to satisfy Definition 3.2.4 (i) and 3.2.4 (ii). We will additionally retain the reference identities $1_y$ and $1_x$ of the original equivalence for Definition 3.2.4 (iii), and so it remains to give a equivalences $1_y \sim \beta \circ \alpha$ and $\alpha \circ \beta \sim 1_x$. However, we have equivalences $\beta \circ \alpha \sim 1_y$ and $1_x \sim \alpha \circ \beta$ by assumption, and so we may apply our construction thus far coinductively to give the desired result. ■

Now let us attend to the transitivity implication of equivalences. In order to give this proof, we will simultaneously develop the proof of a supporting lemma. Although we therefore give a proof by mutual coinduction, each step is "productive" in that data in the appropriate dimension is produced and the mutual appeal to the other result also produces data in the dimension in question.

**Lemma 3.2.13**. *In a proof-relevant category, both of the following statements hold.*

*(i) If there are equivalences $x \sim y$ and $y \sim z$ then there is an equivalence $x \sim z$.*

*(ii) Suppose there are parallel cells $\theta, \varphi$ of arbitrarily high dimension over the boundary $x \leftarrow y$, as well as an equivalence $\theta \sim \varphi$. Then for any cells $w \xleftarrow{f} x$ and $y \xleftarrow{g} z$ as well as any evaluations*

$$
\begin{array}{cc}
\multicolumn{2}{c}{f\theta g} \\ \hline
f\psi_{n-1}g & f\xi_{n-1}g \\ \hline
\vdots & \vdots \\ \hline
w & z
\end{array}
\quad
\overset{\begin{array}{cc}
\multicolumn{2}{c}{a} \\ \hline
r_{n-1} & s_{n-1} \\ \hline
\vdots & \vdots \\ \hline
w & z
\end{array}}{\Longleftarrow}
\quad
\begin{array}{c|c|c}
 & \theta & \\ \hline
 & \psi_{n-1} \quad \xi_{n-1} & \\ \hline
 & \vdots \quad \vdots & \\ \hline
f & \psi_1 \quad \xi_1 & g \\ \hline
w \; x & & y \; z
\end{array}
$$

$$
\begin{array}{cc}
\multicolumn{2}{c}{f\varphi g} \\ \hline
f\psi_{n-1}g & f\xi_{n-1}g \\ \hline
\vdots & \vdots \\ \hline
w & z
\end{array}
\quad
\overset{\begin{array}{cc}
\multicolumn{2}{c}{b} \\ \hline
r_{n-1} & s_{n-1} \\ \hline
\vdots & \vdots \\ \hline
w & z
\end{array}}{\Longleftarrow}
\quad
\begin{array}{c|c|c}
 & \varphi & \\ \hline
 & \psi_{n-1} \quad \xi_{n-1} & \\ \hline
 & \vdots \quad \vdots & \\ \hline
f & \psi_1 \quad \xi_1 & g \\ \hline
w \; x & & y \; z
\end{array}
$$

*there is an equivalence $(f\theta g) \sim (f\varphi g)$.*

Because the mutual-coinductive structure of the proof of this lemma may be potentially strange, we have structured our proof as follows. First we prove (i) assuming that (ii) is productive. Then we prove (ii) assuming that (i) is productive. Then we prove that these mutual assumptions are sound.

*Proof* (Lemma 3.2.13 (i) assuming Lemma 3.2.13 (ii) is productive). Suppose that there are equivalences $x \sim y$ and $y \sim z$, where the former has mediating cells $x \xleftarrow{\alpha} y$ and $y \xleftarrow{\beta} x$, and the latter has mediating cells $y \xleftarrow{\gamma} z$ and $z \xleftarrow{\delta} y$. From this data we wish to construct the data of an equivalence $x \sim z$.

To begin let us identify the mediating cells of the equivalence-to-be $x \sim z$. From the mediating cells of the previous two equivalences let us form the new evaluations

$$x \xleftarrow{\alpha\gamma} z \quad \overset{\frac{\kappa_1}{x\ \ z}}{\Longleftarrow} \quad x \xleftarrow{\alpha} y \xleftarrow{\gamma} z \qquad \text{and} \qquad z \xleftarrow{\delta\beta} x \quad \overset{\frac{\kappa_1'}{z\ \ x}}{\Longleftarrow} \quad z \xleftarrow{\delta} y \xleftarrow{\beta} x\ ,$$

by appealing to the split contraction. From these let us form the evaluations

$$x \xleftarrow{(\alpha\gamma)(\delta\beta)} x \quad \overset{\frac{\kappa_2}{x\ \ x}}{\Longleftarrow} \quad x \xleftarrow{\alpha\gamma} z \xleftarrow{\delta\beta} x \qquad \text{and} \qquad z \xleftarrow{(\delta\beta)(\alpha\gamma)} z \quad \overset{\frac{\kappa_2'}{z\ \ z}}{\Longleftarrow} \quad z \xleftarrow{\delta\beta} x \xleftarrow{\alpha\gamma} z$$

once more by appealing to the split contraction. Our reference identities here shall be $1_x$ of $x \sim y$ and $1_z$ of $y \sim z$, and so to complete the proof we must supply the outstanding data of equivalences $1_x \sim (\alpha\gamma)(\delta\beta)$ and $(\delta\beta)(\alpha\gamma) \sim 1_z$.

In order to do this we shall argue coinductively. We will shortly give instead the following equivalences,

$$\begin{gathered} 1_x \overset{(1)}{\sim} \alpha\beta \overset{(2)}{\sim} \alpha 1_y'\beta \overset{(3)}{\sim} \alpha(\gamma\delta)\beta \overset{(4)}{\sim} (\alpha\gamma)(\delta\beta) \\ (\delta\beta)(\alpha\gamma) \overset{(4)}{\sim} \delta(\beta\alpha)\gamma \overset{(3)}{\sim} \delta 1_y \gamma \overset{(2)}{\sim} \gamma\delta \overset{(1)}{\sim} 1_z\ , \end{gathered} \tag{3.2.14}$$

to whose data we may apply coinductively the proof of transitivity constructed thus far in order to deduce the desired equivalences.

In the above we have fixed evaluations

$$\begin{gathered} x \xleftarrow{\alpha(\gamma\delta)\beta} x \quad \overset{\frac{\kappa_3}{x\ \ x}}{\Longleftarrow} \quad x \xleftarrow{\alpha} y \xleftarrow{\gamma\delta} y \xleftarrow{\beta} x \\ z \xleftarrow{\delta(\beta\alpha)\gamma} z \quad \overset{\frac{\kappa_3'}{z\ \ z}}{\Longleftarrow} \quad z \xleftarrow{\delta} y \xleftarrow{\beta\alpha} y \xleftarrow{\gamma} z \end{gathered}$$

using the evaluations specified by $x \sim y$ and $y \sim z$. With this, let us describe the equivalences we claimed above.

Both equivalences marked (1) are part of the data of our assumed equivalences $x \sim y$ and $y \sim z$. The equivalences marked (2) arise by Lemma 3.2.9 applied to the identities $1_y'$ and $1_y$ of $y \sim z$ and $x \sim y$ respectively. The equivalences marked (3) arise by an appeal to Lemma 3.2.13 (ii) applied to the equivalences $1_y' \sim \gamma\delta$ and $\beta\alpha \sim 1_y$ from the data of $y \sim z$ and $x \sim y$ respectively.

Finally, to account for the equivalences marked (4) let us argue as follows. Observe that $\alpha(\gamma\delta)\beta$ and $(\alpha\gamma)(\delta\beta)$ are both evaluations of the same pasting, with parallel proofs witnessing as much.

$$x \xleftarrow{\alpha(\gamma\delta)\beta} x \quad \overset{\begin{array}{c}\kappa_3 \,\square\, (\alpha|r|\beta) \\ \hline x \qquad\qquad x\end{array}}{\Longleftarrow\!=\!=\!=\!=} \quad x \xleftarrow{\alpha} y \xleftarrow{\gamma} z \xleftarrow{\delta} y \xleftarrow{\beta} x$$

$$x \xleftarrow{(\alpha\gamma)(\delta\beta)} x \quad \overset{\begin{array}{c}\kappa_2 \,\square\, (\kappa_1|\kappa_1') \\ \hline x \qquad\qquad x\end{array}}{\Longleftarrow\!=\!=\!=\!=} \quad x \xleftarrow{\alpha} y \xleftarrow{\gamma} z \xleftarrow{\delta} y \xleftarrow{\beta} x$$

We have written here $r$ for the chosen proof that $\gamma\delta$ is a composite, as contained in $y \sim z$. Thus by Lemma 3.2.7 there is an equivalence as in the first instance of (4) above.

Similarly we note that $(\delta\beta)(\alpha\gamma)$ and $\delta(\beta\alpha)\gamma$ are two evaluations of the same pasting with parallel proofs, and so the second equivalence marked (4) is obtained by Lemma 3.2.7. This concludes the proof of Lemma 3.2.13 (i) assuming that the proof of Lemma 3.2.13 (ii) is productive. ■

*Proof* (Lemma 3.2.13 (ii) assuming Lemma 3.2.13 (i) is productive). Let us write $\lambda$ and $\nu$ for the mediating cells of $\theta \sim \varphi$ and fix first the new mediating cells $f\lambda g$ and $f\nu g$ to be given by the following evaluations where we have used the split contraction.

$$\begin{array}{cc} \multicolumn{2}{c}{f\lambda g} \\ \hline f\theta g & f\varphi g \\ \hline \vdots & \vdots \\ \hline w & z \end{array} \quad \overset{\begin{array}{c}\kappa \\ \hline a\ \ b \\ \vdots\ \ \vdots \\ \hline w\ \ z\end{array}}{\Longleftarrow} \quad \begin{array}{c|cc|c} & \multicolumn{2}{c|}{\lambda} & \\ \hline & \theta & \varphi & \\ \hline & \vdots & \vdots & \\ \hline f & \psi_1 & \xi_1 & g \\ \hline w\ \ x & & & y\ \ z \end{array}$$

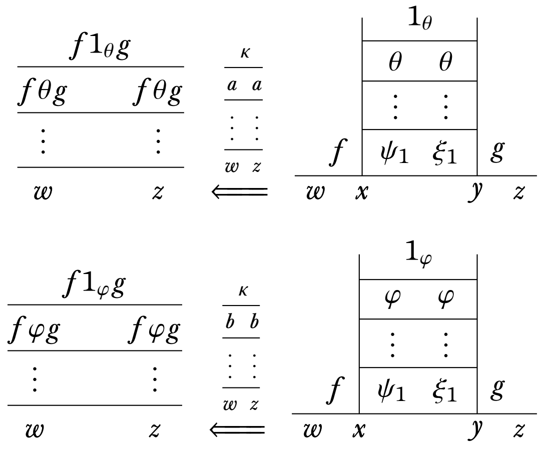
$$\begin{array}{cc} \multicolumn{2}{c}{f\nu g} \\ \hline f\varphi g & f\theta g \\ \hline \vdots & \vdots \\ \hline w & z \end{array} \quad \overset{\begin{array}{c}\kappa \\ \hline b\ \ a \\ \vdots\ \ \vdots \\ \hline w\ \ z\end{array}}{\Longleftarrow} \quad \begin{array}{c|cc|c} & \multicolumn{2}{c|}{\nu} & \\ \hline & \varphi & \theta & \\ \hline & \vdots & \vdots & \\ \hline f & \psi_1 & \xi_1 & g \\ \hline w\ \ x & & & y\ \ z \end{array}$$

As usual we appeal to the split contraction to form evaluations $(f\lambda g)(f\nu g)$ and $(f\nu g)(f\lambda g)$. Next to supply the reference identities let us appeal to Lemma 3.2.3 to obtain $1_{f\theta g}$ and $1_{f\varphi g}$. To complete the construction we must give the data of equivalences $1_{f\theta g} \sim (f\lambda g)(f\nu g)$ and $(f\nu g)(f\lambda g) \sim 1_{f\varphi g}$.

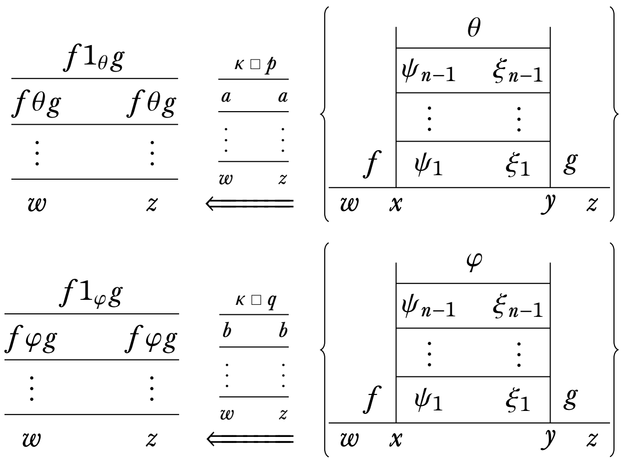
We will not supply these directly, and instead give chains of equivalences to which we shall apply Lemma 3.2.13 (i)—itself shortly to be constructed. We intend to give the following equivalences.

$$\begin{gathered} 1_{f\theta g} \overset{(1)}{\sim} f1_\theta g \overset{(2)}{\sim} f(\lambda\nu)g \overset{(3)}{\sim} (f\lambda g)(f\nu g) \\ (f\nu g)(f\theta g) \overset{(3)}{\sim} f(\nu\theta)g \overset{(2)}{\sim} f1_\varphi g \overset{(1)}{\sim} 1_{f\varphi g} \end{gathered} \tag{3.2.15}$$

To give meaning to these, let us begin by fixing evaluations $f1_\theta g$ and $f1_\varphi g$ as below where $1_\theta$ and $1_\varphi$ are the reference identities of $\theta \sim \varphi$.




Should we write $p$ and $q$ for the proofs that $1_\theta$ and $1_\varphi$ are identities, then we see that proof composition witnesses the above evaluations as evaluations of the degenerate $n$-pastings below.




By proof composition, the identities $1_{f\theta g}$ and $1_{f\varphi g}$ are evaluations of the upper and lower degenerate $n$-cells above, respectively. Moreover, the resulting pair of proofs generated in this way are parallel with their corresponding proof above, and so Lemma 3.2.7 gives the equivalences marked (1) in (3.2.15).

To account for the equivalences marked (3) in (3.2.15) let us fix evaluations $f(\lambda\nu)g$ and $f(\nu\lambda)g$ by the split contraction as usual, where the cells $\lambda\nu$ and $\nu\lambda$ are the evaluations contained in the data $\theta \sim \varphi$. Proof composition shows that $(f\lambda g)(f\nu g)$ and $f(\lambda\nu)g$ are evaluations of the same pasting involving $\lambda$ and $\nu$ whiskered by $f$ and $g$, and moreover the proofs witnessing these evaluations are parallel. Thus Lemma 3.2.7 gives the first equivalence marked (3) in (3.2.15). Entirely similar arguments, but with the orders of $\lambda$ and $\nu$ interchanged give the other equivalence also marked (3).

It remains then to supply the data of the equivalences marked (2). For this purpose, we apply the the proof we have given thus far coinductively to the equivalences $1_\theta \sim \lambda\nu$ and $\nu\lambda \sim 1_\varphi$ contained in the data $\theta \sim \varphi$. ■

We have now given proofs for the two parts of Lemma 3.2.13, coinductively, and in terms of one another. It remains then to convince the reader that the overall construction is sound.

*Proof* (Soundness of mutual assumptions for Lemma 3.2.13). There are two important facets of the above proof that allow us to conclude that the result is sound. First note that whenever a coinductive appeal to the result under construction is made, it is at a strictly higher dimension than the cells currently being considered. In this way the proof is never “stuck”.

Second is a more subtle property, that of productivity. Observe that the proof of Lemma 3.2.13 (i) produces mediating cells, evaluations of their pasting, and reference identities without any complications. Then, of the four types of equivalences displayed in (3.2.14) only the equivalences marked (3) are potentially problematic for those marked (1), (2), and (4) are separate results. The reason we should carefully consider (3) is because this is the appeal to Lemma 3.2.13 (ii).

What is needed to apply Lemma 3.2.13 (i) coinductively to the four displayed equivalences is at first only the data of mediating cells, evaluations of their pastings, and reference identities. Should we trace out the proof of Lemma 3.2.13 (ii) on the pairs of equivalences demanded by (3.2.14), we see that the mediating cells, evaluations of their pastings, and reference identities are immediately produced. Thus we may apply Lemma 3.2.13 (i) coinductively to the data of (3.2.14).

A similar argument, but from the perspective of Lemma 3.2.13 (ii) shows that whenever we need to extract mediating cells, evaluations of their pastings, and reference identities they are always present. Overall then, the proof by mutual coinduction is sound. ■

Now that we have seen some of the basic facets of the theory of equivalences and identities in a proof-relevant category let us turn to some applications.

**Lemma 3.2.16**. *Let $x$ and $y$ be parallel cells of the same dimension, and let us write $\{x \Leftarrow y\}$ for the set of proofs over reflexivity that $x$ is an evaluation of $y$. There is a function taking such proofs to equivalences,* $\mathsf{prftoeqv}\colon \{x \Leftarrow y\} \to x \sim y$.

*Proof.* Let us fix a proof $x \overset{p}{\Leftarrow} y$ over reflexivity, and observe that we therefore a pair of parallel proofs $x \overset{p}{\Leftarrow} y$ and $y \overset{y}{\Leftarrow} y$ of the same pasting. Lemma 3.2.7 gives the desired result. ■

*Remark* 3.2.17. In particular in Lemma 3.2.16 applies to all objects, for 0-cells are all parallel and there can be no lower proof data. That is, whenever there is a proof $x \Leftarrow y$ that some object $y$ evaluates to another object $x$ we have an equivalence $x \sim y$. In this sense, objects proofs are *presented* equivalences between objects—there may be other equivalences which naturally exist between objects, but they are not presented by the proof structure of the proof-relevant category. Indeed, in the extreme case of a 1-category embedded discretely as a proof-relevant category, the only presented equivalences are identities (see Remark 3.3.2).

Unlike ordinary equivalences, however, presented equivalences are not a symmetric notion. There is nothing in the axioms of a proof-relevant category that demands that a proof $x \Leftarrow y$ be paired with a corresponding proof $y \Leftarrow x$. Where they suffer for this, presented equivalences enjoy a more natural compatibility with proof-relevant functors. As we shall see in Lemma 4.1.5, whenever we have a presented equivalence $x \Leftarrow y$ any value obtained by a proof-relevant functor on $x$ is also a value of the functor on $y$. Thus should we have a pair of cells $x$ and $y$ with presented equivalences in both directions, proof-relevant functors all obtain precisely the *same* values on $x$ and $y$. In this way, proofs between objects have a sensible interpretation—they are the proof-relevant category expressing the idea that the objects are the same in the structuralist sense.

This should be understood in contrast to ordinary equivalences in a proof-relevant category. As we shall see in Lemma 4.1.4, proof-relevant functors map equivalences to equivalences and so in this sense proof-relevant functors obtain equivalent values on equivalent cells—but nothing stronger. ◁

We now return to a point raised in Remark 3.2.6 which discusses our definition of equivalences, Definition 3.2.4.

**Lemma 3.2.18.** *Let us temporarily say that an equivalence $x \sim y$ is constrained if all of the reference identities used are the identities of Lemma 3.2.3, and write in this case $x \sim_c y$. An equivalence $x \sim y$ determines the data of a constrained equivalence $x \sim_c y$. Therefore equivalences and constrained equivalences are logically equivalent.*

*Proof.* To "constrain" a given equivalence $x \sim y$, it is enough to replace its reference identities with those of Lemma 3.2.3 and extract from the data $1_x \sim \alpha\beta$ and $\beta\alpha \sim 1_y$ equivalences $1^c_x \sim \alpha\beta$ and $\beta\alpha \sim 1^c_y$ where $1^c$ indicates the identity of Lemma 3.2.3. The reason that this is sufficient is that we may thereafter repeat the "constraining" construction coinductively.

Observe that $1_x$ and $1^c_x$ are both evaluations of the same (degenerate) pasting whose proofs are parallel by virtue of being identities. Thus by Lemma 3.2.7 we have $1^c_x \sim 1_x$. Finally, Lemma 3.2.13 (i) allows us to extract from this and the assumption $1_x \sim \alpha\beta$ the desired equivalence data $1^c_x \sim \alpha\beta$. The argument for the $1^c_y$ is entirely similar. ■

**Conjecture 3.2.19.** *Let us temporarily say that a loose equivalence $x \sim_\ell y$ comprises data similar to Definition 3.2.4 but where the evaluations of the pastings of mediating cells are not required to be presented by proofs over reflexivity. Instead we replace Definition 3.2.4 (ii) by the data of*

*(ii') a pair of evaluations*

$$x \xleftarrow{\alpha\beta} x \quad \overset{\frac{\rho}{p\ q}}{\Longleftarrow} \quad x \xleftarrow{\alpha} y \xleftarrow{\beta} x \qquad \textit{and} \qquad y \xleftarrow{\beta\alpha} y \quad \overset{\frac{\sigma}{q\ p}}{\Longleftarrow} \quad y \xleftarrow{\beta} x \xleftarrow{\alpha} y$$

*where $p$ and $q$ are over reflexivity.*

*Naturally we require that Definition 3.2.4 (iv) is replaced by asking for further loose equivalences $1_x \sim_\ell \alpha\beta$ and $\beta\alpha \sim_\ell 1_y$.*

*Any loose equivalence $x \sim_\ell y$ determines an equivalence $x \sim y$, and therefore loose equivalences and equivalences are logically equivalent.*

*Proof* (Suggestion). By Lemma 3.2.7, observe that $p$ and $\mathsf{refl}_x$ generate an equivalence $x \sim x$, and similarly $q$ and $\mathsf{refl}_y$ generate an equivalence $y \sim y$. The goal is to use these equivalences to "untwist" $\alpha$ and $\beta$ so that the we may exchange the proofs $p$ and $q$ for reflexivity. More specifically, let us write $\gamma_p$ and $\gamma_q$ for the mediating cells of the equivalences arising as

$$x \xleftarrow{\gamma_p} x \xLeftarrow{\frac{\kappa}{x\ p}} \{x\} \qquad \text{and} \qquad y \xleftarrow{\gamma_q} y \xLeftarrow{\frac{\kappa}{y\ q}} \{y\}\ .$$

Let us also suggestively write $\gamma_p^{-1}$ and $\gamma_q^{-1}$ for the mediating cells in the other direction, arising in a symmetric manner to those above. As suggested by our notation, we claim that there are equivalences and a loose equivalence as indicated below.

$$(\gamma_p\alpha\gamma_q^{-1})(\gamma_q\beta\gamma_p^{-1}) \overset{(1)}{\sim} \gamma_p(\alpha\beta)\gamma_p^{-1} \overset{(2)}{\sim}_\ell \gamma_p 1_x \gamma_p^{-1} \overset{(3)}{\sim} 1_x$$

The equivalences numbered (1) and (3) are expected to be formal, while the loose equivalence marked (2) ought to be the result of something like Lemma 3.2.13 (ii) for loose equivalences. In this way, should we choose $\gamma_p\alpha\gamma_q^{-1}$ and $\gamma_q\beta\gamma_p^{-1}$ as our new mediating cells, by Lemma 3.2.13 (i) and a coinductive untwisting of (2) we may derive the desired data of an ordinary equivalence $x \sim y$. ∎

*Remark* 3.2.20. The difference between these loose equivalences and ordinary equivalences is that, in some sense, we have allowed one layer of non-triviality in the proofs witnessing the evaluation of the pasting of the mediating cells. One might image an "entirely loose" equivalence allowing for non-triviality in all dimensions of the proofs. Should the above suggestion be true, then it is reasonable to hope that some inductive process may eventually yield the logical equivalence of "entirely loose" equivalences with equivalences. ◁

Finally let us attend to a technical matter which we mentioned in the introduction. Our definition of proof-relevant category introduces something of a redundancy.

*Remark* 3.2.21. Recall that by Lemma 2.4.3, because $d\colon P \to TA$ is equipped with the data of a split contraction there is a section $s\colon TA \to P$ of $d$. However, as we observed, $\mathsf{refl}$ is also a section of $d$ on 0-cells and so the proof of Lemma 2.4.3 produces a second section $s'\colon TA \to P$ of $d$.

There is no general reason that these sections must coincide, and so we may detect two slightly different behaviours. On 0-cells, $\mathsf{refl}$ is additionally a section of $c\colon P \to A$.

This is not necessarily the case for $s$, and so we have no control over the codomain:

$$x \xLeftarrow{s'(x)} x \qquad \text{vs} \qquad c(s(x)) \xLeftarrow{s(x)} x \,.$$

However, these proofs are parallel for 0-cells so that by Lemma 3.2.7 there is an equivalence $x \sim c(s(x))$. Likewise we suggest that in higher dimensions, by composing with appropriate equivalences in lower dimensions, the results of either section may be equivalent to the other. ◃

## 3.3. Examples

Before we give any explicit examples, let us set expectations with the following lemma.

**Lemma 3.3.1.** *If a proof-relevant category has any cells, it has cells and proofs in all dimensions.*

*Proof.* The idea of this claim is straightforward: if there is a 0-cell then there must be iterated identity cells on it in all dimensions. To make this precise, we argue as follows.

If there is a cell there must be a 0-cell, so let us predicate our arguments thus. Given a 0-cell $a \in A_0$ there are cells in arbitrary dimensions of $TA$, viz., steadily further degenerated copies of $a$. By Lemma 2.4.3, there is a section $s$ of $d\colon P \to TA$ so that there is at least one proof in every dimension which proves that each degenerated version of $a$ evaluates to a cell in the same dimension. Ergo there are cells and proofs in all dimensions. ■

Thus we know to expect that all of our (interesting) examples will have cells and proofs in every dimension. However we can still get away with making almost all of these data trivial so as to recover "discrete" embeddings of finite-dimensional structures.

### 3.3.1 Strict categories

Our first class of examples is the most classical, viz., ordinary strict $n$-categories for $n \in \mathbb{N} \cup \{\infty\}$.

**Algebras of $T$** From more general considerations (see later Lemma 5.3.1) we see that it is possible view algebras of $T$—that is $\omega$-categories—as proof-relevant categories. Nevertheless we can realise this fact in elementary terms here.

Suppose that $(A, h\colon TA \to A)$ is an $\omega$-category, then by Theorem 2.2.11 the following span has the structure of $T$-category.

$$A \xleftarrow{h} TA = TA$$

By Lemma 2.4.5 (i), or as may be seen directly, the identity $TA = TA$ is certainly canonically equipped with the data of a split contraction. Thus the above span is canonically a proof-relevant category, but of an especially degenerate sort.

As we might imagine from strict structures, there is exactly one proof that any given pasting has an evaluation—in this case the proofs are named precisely by the pastings. For this reason, evaluation obeys all the strict coherences and there is exactly one identity for each cell—the usual identity.

Note that $A$ was in fact an $n$-category for some $n$, so that all of its cells in dimension $m > n$ are identities, then we may obtain a substantial simplification in the notion of equivalence. In this more familiar setting, Definition 3.2.4 specialises to the familiar notion of equivalence in an $n$-category: a pair of opposed mediating cells, between whose composites and the identities is another equivalence, all the way up to dimension $n$ at which point the top-dimensional mediating cells are required to be genuinely invertible. Thus, if $A$ was a 1-category then an equivalence is an isomorphism. If $A$ was a 2-category then an equivalence is an ordinary 2-categorical equivalence, and so on.

*Remark* 3.3.2. Recall our notion of a presented equivalence from Remark 3.2.17: a pair of 0-cells $x$, $y$, and a proof $x \Leftarrow y$. In this class of examples the only presented equivalences may be seen to be the identities, for there is but one proof for evaluating $y$, namely $y$ itself. However, there may certainly be many other non-trivial equivalences in the proof-relevant category. ◁

### 3.3.2 Low dimensional examples

This last remark, on the failure of presented equivalences to capture anything non-trivial, suggests that this embedding of strict $n$-categories into proof-relevant categories is perhaps somewhat lacking. While we will not suggest an alternative here, we will note that 1-category-like objects may be recovered as special types of proof-relevant categories.

**Definition 3.3.3.** A proof-relevant category $\mathbb{A}$ is $k$-**truncated** if, whenever $n > k$ there is precisely one $(n+1)$-cell between a pair $\alpha \leftarrow \beta$ of $n$-cells if $\alpha = \beta$, and none otherwise. ◁

Although we took great pains in the previous section to describe how composition in a proof-relevant category is weak, in this case we find something perhaps surprising.

Consider a composable triple of 1-cells $w \xleftarrow{f} x \xleftarrow{g} y \xleftarrow{h} z$ in a 1-truncated proof-relevant category. Let us consider any two associated compositions of this triple over reflexivity proofs.

**Binary compositions of $w \xleftarrow{f} x \xleftarrow{g} y \xleftarrow{h} z$**

| Left associated | Right associated |
|---|---|
| $w \xleftarrow{fg} y \quad \overset{\frac{p_0}{w\;\;y}}{\Longleftarrow} \quad w \xleftarrow{f} x \xleftarrow{g} y$ | $x \xleftarrow{gh} z \quad \overset{\frac{q_0}{x\;\;z}}{\Longleftarrow} \quad x \xleftarrow{g} y \xleftarrow{h} z$ |
| $x \xleftarrow{(fg)h} z \quad \overset{\frac{p_1}{x\;\;z}}{\Longleftarrow} \quad w \xleftarrow{fg} y \xleftarrow{h} z$ | $x \xleftarrow{f(gh)} z \quad \overset{\frac{q_1}{w\;\;z}}{\Longleftarrow} \quad w \xleftarrow{f} x \xleftarrow{gh} z$ |

Because the proofs $p_1$ and $q_1$ are parallel, we may apply the data of the split contraction to find at least one proof that we may evaluate the following degenerate pasting diagram.

$$\begin{tikzcd} w & z \arrow[l, bend right=40, "(fg)h"'] \arrow[l, bend left=40, "f(gh)"] \arrow[l, phantom, "*\Uparrow"] \end{tikzcd} \quad \overset{\frac{\kappa}{\frac{p_1\;\;q_1}{w\;\;z}}}{\Longleftarrow} \quad \{w \xleftarrow{f} x \xleftarrow{g} y \xleftarrow{h} z\}$$

However, $\mathbb{A}$ is 1-truncated so the 2-cell marked $*$ which must be present as the 2-cell $* = c(\kappa)$ forces the equality $(fg)h = f(gh)$! In this way, binary composition over reflexivity is strictly associative and may be seen to be strictly unital too, but there still may in general be many proofs which show that $(fg)h = f(gh)$. When the binary composition is not over reflexivity, as we have seen in Remark 3.2.17, these object proofs introduce equivalences—here isomorphisms—with which we may conjugate to relate such evaluations to ones whose proofs are over reflexivity. As such, we suggest that a 1-truncated proof-relevant category is akin to an "ana-category": a category in which there may be many results for composition, but for which all of the results are related by isomorphisms of their domains and codomains, and by making choices we may recover strictly unital and associative composition.

In a similar vein, were $\mathbb{A}$ to be 2-truncated instead, then we would see that binary composition over reflexivity is not strictly associative, but instead associative up to suitably natural 2-isomorphisms. That is, we will have found not a category, but something like a bicategory instead. However, once again, there would be in general many proofs which witness the invertibility of these 2-isomorphisms and moreover potentially many 2-isomorphisms and accompanying proofs which prove the bicategory-like associativity of 1-cell composition.

These examples suggest that it is possible to recover traditional weak categorical structures from proof-relevant categories without the need to suppress interesting proof data.

### 3.3.3 Algebras of contractible globular operads

The general theory of globular operads guarantees a plentiful supply of proof-relevant categories.

**Higher categories in the sense of Leinster** In [Lei09, §9.2], a compelling case is made for a definition of weak $\omega$-category in terms of algebras for a certain contractible globular operad $L$. As we shall see in Lemma 5.3.5, each such algebra is a canonically a proof-relevant category.

In particular this is true of [Lei09, Ex. 9.2.7], there is an algebra for $L$ called the fundamental $\omega$-groupoid $\Pi_\omega X$ for each topological space $X$, and thus these are among the proof-relevant categories.

**Classical monoidal categories** A classical monoidal category $(\mathcal{M}, \otimes, I)$ may be viewed as a proof-relevant category. As we begin our elaboration, note that this process is merely a recasting of data rather than a proof-relevant model of monoidality as we had discussed in the introduction—such a class of examples we treat in Section 3.3.4 below.

Given $(\mathcal{M}, \otimes, I)$, let us say that there is a unique 0-cell, that the 1-cells are the objects of $\mathcal{M}$ and that the 2-cells are the morphisms of $\mathcal{M}$. Thereafter let us add cells in higher dimensions discretely: only when the source and target are equal is there a unique cell.

Now let us describe the proofs. In dimension zero let us say that there is a unique proof interpreting the unique 0-cell to itself. Given a pasting of 1-cells, that is an arbitrary string of objects of $\mathcal{M}$, a proof over that string should be a chosen binary bracketing of a generic string of that length. If that string has length zero, that is the pasting is degenerate on the unique object, then let us declare that there is a unique empty proof. If the string has length one, then we declare that there is a unique unary proof. The cell which results as an evaluation for a given proof is the tensor of $\mathcal{M}$ applied in accordance with the chosen bracketing, or $I$ in the nullary case.

Typical of this situation therefore is the following

$$* \xleftarrow{A((BC)D)} * \quad \overset{\frac{\cdot((\cdot)\cdot)}{*\qquad *}}{\Longleftarrow} \quad * \xleftarrow{A} * \xleftarrow{B} * \xleftarrow{C} * \xleftarrow{D} * .$$

We then use the 2-dimensional proofs to encode all of the coherences of the monoidal category as follows: whenever there are two bracketings of equally many terms, there is a unique proof between them. Proof composition therefore performs substitution of bracketing, and from this we recover the rest of the structure of the monoidal category. To match the discrete higher cells, we take proofs to be discrete in dimensions beyond this.

**Classical bicategories** Given the above description, we might imagine that an entirely similar construction for bicategories is possible. For a bicategory $\mathcal{K}$ viewed as a proof-relevant category, the difference in construction is essentially a dimension shift. The 0-cells of the proof-relevant category are the objects of $\mathcal{K}$, the 1-cells are the morphisms, and the 2-cells are the 2-morphisms. Higher cells are once more discrete.

As above, the proofs in dimension one are similarly binary bracketings of a generic string of a given length, and the proofs in dimension two are added uniquely based on matching lengths. However, this time we must attend to object proofs: we add a unique proof for each object as we do not wish to introduce any additional equivalences into the bicategory.

**The general principle** As we have alluded to, at play here is a general principle: algebras for any contractible globular operad in the sense of [Lei09, §9.2] canonically bear the structure of proof-relevant categories. We will see this statement more generally in Lemma 5.3.5, but for now the above examples were generated by looking at $L$, the initial contractible globular operad, the operad for classical monoidal categories, and the operad for classical bicategories. The reader is invited to verify that the latter two listed are contractible. Further examples along these lines include the so-called unbiased variants of monoidal categories and bicategories, those structures which specify one composition operation for each $n$, as well as all other 'plausible'[4] signatures for these as outlined in [Lei09, §3.2]. Variations on this theme yield flavours of weaker higher categories too, such as tricategories of a certain nature realised as algebras for a contractible globular operad and so on.

### 3.3.4 Composition defined by universal properties

In giving the previous examples we have drawn either on the extant theory of contractible globular operads, or looked at especially degenerate proof-relevant categories. Now we wish to give examples which we hope help to justify the implicit promise in the nomenclature: proof-relevant categories where the proofs truly do carry relevant information.

The general theme here will be that, in contrast to other weak category structures where certain cells carry universal factorisation properties but not the data which enables those factorisations, in proof-relevant categories we will not have to compromise in this way. This is perhaps made clearest by example.

**Tensor products of vector spaces** For our first example let us be as concrete as possible. Consider the usual category $\mathrm{FDVect}_k$ of finite dimensional vector spaces over a fixed field $k$. In line with Criteria 1, 2 and 3 of the introduction, we wish to construct a proof-relevant monoidal category $\mathbb{FDVect}_k$ in which tensor is multi-valued, unbiased, and carries proof data.

In keeping with viewing tensor as composition, we engage in a dimensional shift: there is a unique 0-cell of $\mathbb{FDVect}_k$, and it is the 1- and 2-cells that are the finite dimensional vector spaces over $k$ and linear maps respectively. Thereafter we take only discrete higher cells.

[4]Plausability of a signature, as defined by Leinster, amounts to contractibility of the operad it generates.

Let us turn our attention to the proofs. We have no interest in the unique 0-cell $*$ in this monoidal context and so we add only a single proof $* \Leftarrow *$. In order to state the 1-dimensional proofs we must first examine 1-dimensional pastings. Such a pasting $\pi$ of 1-cells is an ordered, finite, potentially empty list $\pi = [V_1, \dots, V_n]$ of finite dimensional vector spaces. With this, we say that a proof $p$ that a vector space $W$ is an evaluation of such a pasting $\pi$ is a natural bijection between the functor of $n$-multilinear maps $\mathrm{Multi}(V_1, \dots, V_n; -) \colon \mathrm{FDVect}_k \to \mathrm{Set}$ and the represented functor $\mathrm{FDVect}_k(W, -) \colon \mathrm{FDVect}_k \to \mathrm{Set}$. Thus we see that proofs are genuinely data external to the ordinary category $\mathrm{FDVect}_k$, and in particular are not of the same sort as the cells of $\mathbb{F}\mathrm{DVect}_k$. Proof composition for such proofs amount to composition of natural bijections and an exercise in linear algebra, and the reflexivity proofs are implemented by the identification $\mathrm{Multi}(V; -) = \mathrm{FDVect}_k(V, -)$. Finally there is enough data to construct the split contraction, one may always construct a tensor product as a quotient of the cartesian product.

Now let us examine 2-dimensional pastings so as to understand the 2-dimensional proofs. Given parallel 1-dimensional pastings $\pi = [V_1, \dots, V_n]$ and $\pi' = [V_1', \dots, V_n']$ a 2-dimensional pasting $\Pi$ from $\pi'$ to $\pi$ is a list of linear maps $[f_1, \dots, f_n]$ with each map being of the form $f_i \colon V_i' \to V_i$. Given such a 2-dimensional pasting $\Pi$, as well as two proofs

$$W \overset{p}{\Leftarrow} [V_1, \dots, V_n] \qquad \text{and} \qquad W' \overset{p'}{\Leftarrow} [V_1', \dots, V_n'] \ ,$$

a proof $g \overset{\rho}{\Leftarrow} \Pi$ where $g \colon W' \to W$ is a linear map is a proof that the below square is commutative. As this is a property, proofs in dimension 2 are unique when they exist.

$$\begin{tikzcd}
\mathrm{Multi}(V_1, \dots, V_n; -) \arrow[r, "p'"] \arrow[d, "\mathrm{Multi}(f_1, \dots, f_n; -)"'] & \mathrm{FDVect}_k(W, -) \arrow[d, "\mathrm{FDVect}_k(g, -)"] \\
\mathrm{Multi}(V_1', \dots, V_n'; -) \arrow[r, "p"'] & \mathrm{FDVect}_k(W', -)
\end{tikzcd}$$

Although more complicated to state in elementary terms, proof composition in this dimension is given essentially by nesting and composition of commutative squares and amounts to straightforward considerations of linear algebra. Reflexivity proofs are those squares whose left and right vertical maps are identities, once more under the identification $\mathrm{Multi}(V; -) = \mathrm{FDVect}_k(V, -)$. The split contraction is derived from a Yoneda-like argument: given only the top, left, and bottom morphisms of the square above we may evaluate at $W$ and trace the identity linear map $W \to W$ under $p^{-1}$, pre-composition by the $f_i$, and $p'$ to find a suitable evaluation $W' \to W$. The reader is invited to provide the remaining structure in higher dimensions and verify that all the relevant laws hold.

**Spans in a category** Let us fix a category $\mathscr{C}$ in which limits over zig-zags exist—including in particular pullbacks.[5] We would like to recover something like the bicategory Span $\mathscr{C}$ of spans but without having to restrict ourselves to a single pullback for each cospan and without losing the information that composites are in fact limiting cones. To that end let us describe a proof-relevant category $\mathbb{S}$pan $\mathscr{C}$.

The 0-cells of $\mathbb{S}$pan $\mathscr{C}$ are the objects of $\mathscr{C}$, the 1-cells are the spans of $\mathscr{C}$, and the 2-cells are morphisms of spans of $\mathscr{C}$. We have no interest in higher cells, and so add them discretely.

Now let us describe the proofs. For simplicity let us take a unique zero-dimensional proof for each object so that reflexivity and proof composition in this dimension are clear. Now let us suppose that we had a pasting $\pi$ of 1-cells, that is, a zig-zag as follows.

$$\pi :\equiv \boxed{x_1 \xleftarrow{f_1} y_1 \xrightarrow{g_1} x_2 \xleftarrow{f_2} y_2 \xrightarrow{g_2} \cdots \xleftarrow{f_n} y_n \xrightarrow{g_n} x_{n+1}}$$

If $n = 0$ then the string of spans is an object $x_0$, and otherwise it is as we have displayed. We say that a proof that this string evaluates to a cell,

$$x_0 \xleftarrow{\lambda_0} L \xrightarrow{\lambda_{n+1}} x_{n+1} \quad \xLeftarrow{\frac{(L,\lambda)}{x_0 \qquad x_{n+1}}} \quad x_1 \xleftarrow{f_1} y_1 \xrightarrow{g_1} x_2 \xleftarrow{f_2} y_2 \xrightarrow{g_2} \cdots \xleftarrow{f_n} y_n \xrightarrow{g_n} x_{n+1} ,$$

comprises the data of a cone $(L,\lambda)$ over the diagram $\pi$ and a *proof* that it is limiting, where $c(L,\lambda)$ is the span we have displayed above. Here then proofs are genuinely additional data that is exterior to the bicategory: both the cone and the proof of its universal property are not morphisms of $\mathscr{C}$ or spans in any meaningful way.

There is a reflexivity proof for each span, namely the vertex of the span itself is canonically a limit over that span. Moreover, there is enough data to define the split contraction: our assumption on $\mathscr{C}$ was precisely such that we had a way of selecting a limiting cone over each such zig-zag. Proof composition for these proofs takes on the form of lemma generalising pullback pasting: given a limiting cone $(L,\lambda)$ over a zig-zag, each of whose objects $L_i$ is itself a limiting cone $(L_i,\lambda_i)$ over a further zig-zag, composition of cones witnesses $(L,\langle\lambda_i\rangle \circ \lambda)$ as a limiting cone over the overall concatenation of zig-zags.

Finally in dimension two we take a proof $(u,\{u_i\})$ from $(L',\lambda')$ to $(L,\lambda)$ to be the data of a morphism $u\colon L' \to L$ of $\mathscr{C}$ paired with a family of span morphisms $\{u_i\}$ as indicated below, such that $u$ is the unique morphism between limiting cones corresponding to the induced diagram map.

$$\begin{tikzcd}
 & y_1 \arrow[dl, "f_0"'] \arrow[dr, "g_0"] & & y_2 \arrow[dl, "f_1"'] & \cdots & y_n \arrow[dr, "g_n"] & \\
x_1 & & x_2 & & \cdots & & x_{n+1} \\
 & y_1' \arrow[ul, "f_1'"] \arrow[ur, "g_1'"'] \arrow[uu, "u_1"] & & y_2' \arrow[ul, "f_2'"] \arrow[uu, "u_2"] & \cdots & y_n' \arrow[ur, "g_n'"'] \arrow[uu, "u_n"] &
\end{tikzcd}$$

[5] We mean this in the algebraic sense: there is a chosen limit for each zig-zag.

The pasting $d(u, \{u_i\})$ of a two dimensional proof is the family that generated it, and we consider a degenerate 1-cell pasting to be a 2-cell pasting as above where each $u_i = \mathrm{id}_{y_i}$. The cell $c(u, \{u_i\})$ is the morphism $u \colon L' \to L$ which is necessarily a span morphism. Proof composition in dimension two is dictated by concatenation of the families of span morphisms, and reflexivity is given by the identity family and hence the identity morphism. The reader is invited to verify that the remaining structure and laws follow suitably naturally.

**Profunctors** In a similar way, one might imagine forming a proof-relevant category $\mathbb{P}\mathrm{ROF}$. Here the 0-cells are categories, the 1-cells are profunctors, and the 2-cells are transformations. Again for simplicity we take a unique proof for each object, so that we may now focus on proofs for 1-cells.

Giving a pasting of 1-cells, that is a composable sequence of profunctors, we declare that a proof that this pasting evaluates to a cell

$$\mathscr{A}_0 \overset{F}{\nrightarrow} \mathscr{A}_n \quad \xLeftarrow{\frac{F}{\mathscr{A}_0 \;\; \mathscr{A}_n}} \quad \mathscr{A}_0 \overset{F_0}{\nrightarrow} \dots \overset{F_n}{\nrightarrow} \mathscr{A}_n$$

comprises the data of a profunctor $F \colon \mathscr{A}_0 \nrightarrow \mathscr{A}_n$ together with extranatural transformations $F_0(-, A_1) \times \cdots \times F_n(A_{n-1}, -) \Rightarrow F(-, -)$ and a proof that they are appropriately universal. The rest of the structure of the proof-relevant category follows an analogous development to $\mathbb{S}\mathrm{pan}\,\mathscr{C}$ above.

One might imagine that suitable adaptions of these constructions may be made whenever composition is defined by some universal property: instead of having to select a composite from each class of universal elements and coerce the result into an algebraically formulated categorical structure, we may instead collect the totality of such objects and separate the proofs from the cells. This enables us not only to preserve the relevant data which proves the universality—the very reason that coherences are hold and that category-like structures are possible—, but also allows us to avoid having to say that such and such cells are special because they stand to indicate some universal property whose details are inaccessible to the structure.

# 4. Proof-relevant functors

*It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation [...]* [EM45]

Our goal for this chapter is to transport the above sentiment to proof-relevant structures. That is, we will aim to define proof-relevant functors in a manner that subsumes the notion of proof-relevant category. To this end proof-relevant functors will exhibit many of the same structural properties of proof-relevant categories: they will be multi-valued, they will have proofs that values are obtained on inputs, and they will be unbiased in a perhaps novel way.

Thus in Section 4.1 we give the definition of a proof-relevant functor between proof relevant categories and guide the reader through understanding the model, through confronting its perhaps unusual features, and finally through proving that certain expected interactions with equivalences and proofs hold. In Section 4.2 we then turn to consider some examples and encounter first the statement of Lemma 4.2.1: every proof-relevant category is canonically a proof-relevant endofunctor. We then use this understanding to relate results on proof-relevant functors to those in Section 3, and to suggest that proof-relevant categories are their own identity-like proof-relevant endofunctors. Finally we examine further examples of proof-relevant functors in an attempt to motivate their features. These are the examples of generalised anafunctors in Section 4.2.3 and those proof-relevant functors arising from universal constructions in Section 4.2.5.

## 4.1. The definition

Let $T$ be the free strict $\omega$-category monad of Section 2.5 on the category $\widehat{\mathbb{G}}$ of globular sets. Additionally let us match notation with Section 3.1. In this context, we make the following definition.

**Definition 4.1.1.** Let $\mathbb{A}$ and $\mathbb{B}$ be proof-relevant categories. A **proof-relevant functor** $F\colon \mathbb{B} \to \mathbb{A}$ comprises the data of a span

$$A \xleftarrow{\tau} F \xrightarrow{\sigma} TB$$

along with 2-morphisms $\square^{\sigma}\colon F \circ \mathbb{B} \to F$ and $\square^{\tau}\colon \mathbb{A} \circ F \to F$ in $\mathrm{Span}(\widehat{\mathbb{G}}, T)$, and the data of a split contraction $\kappa$ on $\sigma$. These data are required to render commutative the following diagrams of 2-morphisms in $\mathrm{Span}(\widehat{\mathbb{G}}, T)$

$$
\begin{tikzcd}
 & F \arrow[dl, "\langle \mathsf{refl}^{\mathbb{A}} \circ \mathrm{id}_F \rangle"'] \arrow[d, equal] \arrow[dr, "\langle \mathrm{id}_F \circ \mathsf{refl}^{\mathbb{B}} \rangle"] & \\
\mathbb{A} \circ F \arrow[r, "\Box^{\tau}"'] & F & F \circ \mathbb{B} \arrow[l, "\Box^{\sigma}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \circ \mathbb{A} \circ F \arrow[r, "\mathrm{id}_{\mathbb{A}} \circ \Box^{\tau}"] \arrow[d, "\mathsf{comp}^{\mathbb{A}} \circ \mathrm{id}_F"'] & \mathbb{A} \circ F \arrow[d, "\Box^{\tau}"] \\
\mathbb{A} \circ F \arrow[r, "\Box^{\tau}"'] & F
\end{tikzcd}
$$

$$
\begin{tikzcd}
F \circ \mathbb{B} \circ \mathbb{B} \arrow[r, "\Box^{\sigma} \circ \mathrm{id}_{\mathbb{B}}"] \arrow[d, "\mathrm{id}_F \circ \mathsf{comp}^{\mathbb{B}}"'] & F \circ \mathbb{B} \arrow[d, "\Box^{\sigma}"] \\
F \circ \mathbb{B} \arrow[r, "\Box^{\sigma}"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \circ F \circ \mathbb{B} \arrow[r, "\Box^{\tau} \circ \mathrm{id}_{\mathbb{B}}"] \arrow[d, "\mathrm{id}_{\mathbb{A}} \circ \Box^{\sigma}"'] & F \circ \mathbb{B} \arrow[d, "\Box^{\sigma}"] \\
\mathbb{A} \circ F \arrow[r, "\Box^{\sigma}"'] & F
\end{tikzcd}
,
$$

where we have suppressed the structural 2-morphisms of $\mathrm{Span}(\widehat{\mathbb{G}}, T)$ which mediate associativity and unitality. ◅

*Remark* 4.1.2. The reader with a keen interest in the apparatus of virtual double categories might recognise the above definition of a proof-relevant functor as that of a bimodule in $\mathbb{K}\mathrm{Mod}(\widehat{\mathbb{G}}, T)$, with the added stipulation that the domain leg $\sigma$ have the data of a split contraction. Indeed while there is a coincidence of such definitions, the inherent asymmetry in $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\widehat{\mathbb{G}}, T)$ leads to the interpretation of such an object as being rather different from traditional notions of bimodules. Moreover, as we shall see later in Section 5.4, and specifically in Example 5.1.8, this definition arises from a broader framework: proof-relevant functors are proof-relevant algebras for the $T$-category shaped by the free arrow.

For this reason we do not take the notion of bimodule as logically prior and instead chose to gave the above, elementary definition. ◅

### 4.1.1 Understanding the model

We will develop our understanding of proof-relevant functors in a similar manner to Section 3.1.2, and we hope that our language will mirror and evoke the explanations therein.

Let us begin by observing that we have a globular morphism $\langle \tau, \sigma \rangle \colon F \to A \times TB$ and so we are permitted to look at the fibres over pairs $(k, \pi) \in (A \times TB)_n$. That is, for an $n$-pasting diagram $\pi$ of cells of $\mathbb{B}$ and an $n$-cell $v$ of $\mathbb{A}$ we may examine the $n$-cells $f \in F$ above $(v, \pi)$ and we shall consider these cells to be *proofs $f$ that the specified $n$-cell $v$ is a value of $F$ on the pasting $\pi$.*

For example, if $\pi$ is the 1-pasting below-right and $k$ is the 1-cell below-left, then a 1-cell $f \in f$ above $(v, \pi)$ should be understood as *a* proof that $\ell$ is *a* value of the pasting of $g, h$, and $k$.

$$
\begin{tikzcd}
 & f \in F_1 \arrow[dl, "\tau"'] \arrow[dr, "\sigma"] & \\
\boxed{Fw \xleftarrow{v} Fz} \in A_1 & & TB_1 \ni \overset{\pi}{\boxed{w \xleftarrow{g} x \xleftarrow{h} y \xleftarrow{k} z}}
\end{tikzcd}
$$

To mirror the notation of proof-relevant categories, but to evoke the notion of mapping instead, we will notate such arrangements with annotated mapping arrows as follows.

$$v \overset{f}{\longleftarrow\!\shortmid} \pi$$

Immediately two aspects of this explanation—that functors may obtain multiple values at all, and that functors obtain values on pastings of cells—will perhaps seem alien, but for the time being we defer such questions to the next section. Let us for now satisfy ourselves by pointing out the other salient features of the model.

We emphasised above that $f$ is *a* proof that $v$ is *a* value. This is because there may in general be many different 1-cells $f \in F$ over the pair $(v, \pi)$, intuitively corresponding to the various ways of mapping $g$, $h$, and $k$, say perhaps first composing and then mapping, or mapping and then composing, or various things in-between or even otherwise. Note that the set of proofs varies not only with the shape of the pasting $\pi$, but also with the cells themselves involved in the pasting and the intended value, $v$.

There is more yet, for $F$ is a globular set. Let us consider the situation in the below schematic illustration.

$$\begin{tikzcd} & & \varphi \in F_2 \arrow[dl, "\tau"'] \arrow[dr, "\sigma"] & & \\ \boxed{\begin{tikzcd} Fw & Fz \arrow[l, bend right, "v"'] \arrow[l, bend left, "v'"] \arrow[l, phantom, "\Uparrow\alpha"] \end{tikzcd}} \in A_2 & & & TB_2 \ni \underset{\Pi}{\boxed{\begin{tikzcd} w & x \arrow[l, "g"'] & y \arrow[l, "h"'] & z \arrow[l, bend right, "k"'] \arrow[l, bend left, "k'"] \arrow[l, phantom, "\Uparrow\beta"] \end{tikzcd}}} \end{tikzcd}$$

As $\varphi \in F_2$ we know that it must have source and target 1-cells of $F_1$, let us call these $f :\equiv \mathsf{target}\,\varphi$ and $f' :\equiv \mathsf{source}\,\varphi$. On our account then, $\varphi$ is a proof with *source and target proofs*. More still, as $\sigma$ and $\tau$ are globular morphisms we see that, for instance, $\sigma f = \sigma\,\mathsf{target}\,\varphi = \mathsf{target}\,\sigma\varphi = \mathsf{target}\,\Pi$ where $\mathsf{target}\,\Pi = \pi$ by inspection. That is, the data of a proof $\varphi \in F_2$ above the pair $(\alpha, \Pi)$ includes also *proofs that the boundaries of* $\alpha$ *are values of the boundaries of* $\Pi$ . In this case the intuition is that to give a proof that $\alpha$ is a value of the whiskering of the 2-cell $\alpha$ with $g$ and $h$ we must first settle how we intend to map the boundaries.

### 4.1.2 A tour of the features

Now let us attend to the perhaps novel subtleties of the definition of proof-relevant functor, which we had previously deferred. To simplify matters as we contend with the first unusual aspect of proof-relevant functors, let us narrow our focus to pastings $\pi \in TB$ which are simply cells $b \in B$. That is, let us consider proof-relevant functors inasmuch as they map cells to cells.

Given a cell $b$ of $\mathbb{B}$ it is possible that in general there will be many distinct values

$$a \overset{f}{\longleftarrow\!\shortmid} b$$

in $\mathbb{A}$, each of which we argue deserves equally to be understood as a value of the functor on the cell $b$. This is perhaps not an entirely foreign concept, for instance the anafunctors of [Mak96] are multi-valued too. In Section 4.2.5 we hope to further motivate that multi-valued-ness is natural arrangement of affairs when considering proof-relevant functors whose values are determined by universal constructions—for example, by taking products. As these instances might suggest, the different values obtained by proof-relevant on a fixed cell are not arbitrary but in fact intimately related. To see this let us turn to the data of the split contraction $\kappa$ on $\sigma\colon F \to TB$.

Recall that by Lemma 2.4.3 there is a splitting of $\sigma$ as a globular morphism. As such, the presence of the split contraction $\kappa$ guarantees that there is *at least one* value and proof for every cell in $\mathbb{B}$. In this way at least proof-relevant functors are required to map. However, there is also in essence only one value up to equivalence. While we will wait for Lemma 4.1.3 below to make a general and rigorous statement in this direction, let us indicate the basic principle involved. Suppose then that on a given 0-cell $b$ we had the two 0-cell values

$$a \xleftarrow{f} b \qquad \text{and} \qquad a' \xleftarrow{f'} b$$

in $\mathbb{B}$. From this data, and by treating $b$ as the degenerate 1-pasting $\{b\} \in (TA)_1$, we may form the following lifting problem for $\sigma$.

$$\begin{tikzcd} \partial_1 \arrow[r, "\langle f, f' \rangle"] \arrow[d, tail] & F \arrow[d, "\sigma"] \\ 2_1 \arrow[r, "\{b\}"'] & TB \end{tikzcd}$$

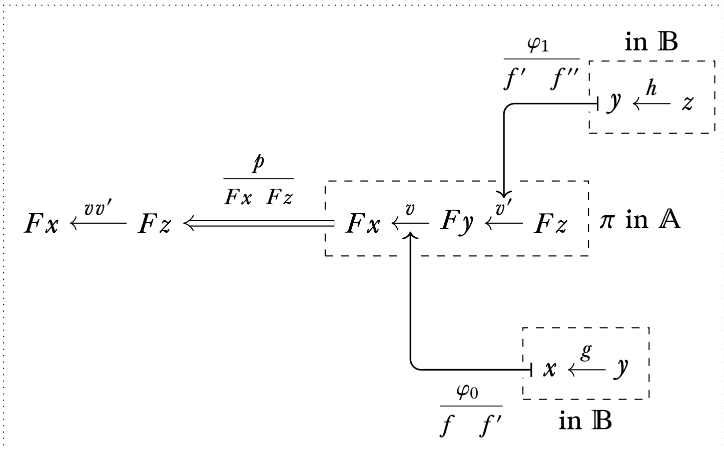
The presence of the split contraction prescribes a solution $\kappa(\{b\}, \langle f, f' \rangle)\colon 2_1 \to F$, that is, a value of $\{b\}$ with a proof over $\langle f, f' \rangle$. There is a symmetry to this situation given by interchanging the roles of $f$ and $f'$ and the net result is that we obtain opposed 1-cells in $\mathbb{A}$ as follows.

$$a \xleftarrow{\alpha} a' \xleftarrow{\frac{\kappa}{f\ f'}} \{b\} \qquad \text{and} \qquad a' \xleftarrow{\beta} a \xleftarrow{\frac{\kappa}{f'\ f}} \{b\}$$

As we might hope, it turns out that these opposed cells are the mediating cells of an equivalence $a \sim a'$ so that the values of $F$ on a fixed 0-cell are all equivalent.

Should we attempt to elaborate this argument with our present understanding of proof-relevant functors, however, we will immediately come up against an issue. How if at all are evaluations of pastings of values obtain by $F$ related to $F$? The dual question presents itself too: how if at all are the values of $F$ on an evaluation of a pasting related to the values of $F$ on each cell? In perhaps more familiar terminology: how are we to understand the notions of mapping-then-composing and composing-before-mapping in the context of a proof-relevant functor?

This is where the action $\square^{\tau}$ of the codomain $\mathbb{A}$ and the action $\square^{\sigma}$ of the domain $\mathbb{B}$ enter consideration. Let us focus on the action of the codomain first, and therefore explicate the globular set $\mathbb{A} \circ F$ which is the domain of the map $\square^\tau$. By computing the relevant pullback to calculate $\mathbb{A} \circ F$ we see that its cells are pairs of evaluations $\alpha \Leftarrow \pi$ in $\mathbb{A}$ and pastings $\Pi$ of proofs of $F$ such that the pasting of values $(T\tau)\Pi$ is $\pi$. We indicate a sample such situation schematically below.



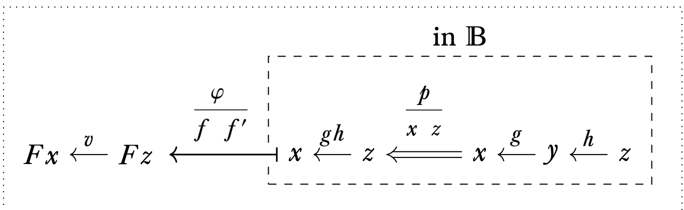
$\in (\mathbb{A} \circ F)_1$

From the example data above, the action of the codomain on $F$ allows us to infer that $vv'$ is a value of on the pasting of the diagrams in $\mathbb{B}$. In this case we therefore have a proof

$$Fx \xleftarrow{vv'} Fz \quad \xleftarrow[\;f \qquad f'\;]{\; p \,\square^\tau\, (\varphi_0|\varphi_1)\;} \quad x \xleftarrow{g} y \xleftarrow{h} z\ .$$

Note that in the above expression we have made of the fact that the proof $p$ was over proofs of reflexivity in $\mathbb{A}$, and $\square^\tau$ is appropriately unital. This then is our first understanding of what it means for a proof-relevant functor to obtain a value on a pasting: among the values on the pasting those values formed by the map-then-compose process.

However, there is also an action $\square^\sigma$ of the domain $\mathbb{B}$ on $F$. Upon computing the pullback $F \circ \mathbb{B}$ which forms the domain of this map, we see that it comprises pairs of values $a \leftarrow\!\dashv \pi$ and pastings $\Pi$ of proofs in $\mathbb{B}$ whose pasting of evaluations $(Tc)\Pi$ is $\pi$. We schematically indicate a particularly simple, sample such situation below.



$\in (F \circ \mathbb{B})_1$

From the example data above, the action of the domain on $F$ allows us to infer that $v$ is a value on the pasting $g|h$ in $\mathbb{B}$. In this case we therefore have a proof

$$Fx \xleftarrow{v} Fz \quad \xleftarrow[\;f \qquad f'\;]{\;\varphi \,\square^\sigma\, p\;} \quad x \xleftarrow{g} y \xleftarrow{h} z\ .$$

Note that in the above expression we have made use of the unitality of $\square^\sigma$ with respect to $\mathsf{refl}^{\mathbb{B}}$ so as to determine that the boundary of $\varphi \,\square^\sigma\, p$ is as we claim. This then is our second understanding of what it means for a proof-relevant functor to obtain a value on a pasting: among the values on the pasting are those values formed by the compose-then-map process.

This concludes the tour of the essential features of a proof-relevant functor, but perhaps leaves one glaring aspect unaddressed. We have been careful so far to say "*among* the values a proof-relevant functor obtains on a pasting", but it would seem that the duo of map-then-compose and compose-then-map (and those in-between combinations of the two) exhausts the imagination for all possible values. Here we must contend with the final subtlety of proof-relevant functors.

Like the proof-relevant categories upon which they are predicated, proof-relevant functors are entirely unbiased. That is, they do not privilege mapping single cells any more than they privilege mapping any other pasting shape. More still, there may be fewer or more values obtained depending on precisely which cells are in the pasting at hand. Comparisons to other models of transfors of higher-category-like structures are strained, for while the idea of unbiased composition in a category seems to be slowly permeating the field[6] it seems that we are still some distance away from the notion of unbiased functor. At the time of writing, this may be one of the few or perhaps even the first instance of such a model.

However, this is not as strange as it may sound, and we hope to convince the reader in Section 4.2.5 that even a careful reading of "the product functor" necessarily produces a proof-relevant functor which obtains values on pastings which do *not* arise from any combination of map-then-compose or compose-then-map. As such we defer further exposition along these lines and turn to outline some cursory aspects of the elementary theory of proof-relevant functors.

First let us establish a general sense in which values of a proof-relevant functor on a fixed pasting are equivalent.

**Lemma 4.1.3.** *Let $F\colon \mathbb{B} \to \mathbb{A}$ be a proof-relevant functor, and let $\pi$ be a pasting in $\mathbb{B}$. If there are two parallel proofs*

$$\alpha \xleftarrow{f} \pi \qquad \textit{and} \qquad \alpha' \xleftarrow{f'} \pi$$

*then there is canonically the data of an equivalence $\alpha \sim \alpha'$ in $\mathbb{A}$.*

*Proof.* This proof proceeds in a similar fashion to that of Lemma 3.2.7, but with some details changed to accommodate the fact we are now working between proof-relevant categories as opposed to in a single one.

As equivalences are defined coinductively, our proof will be structured coinduc-

[6] Through a certain lense, the utility of presented multicategories over monoidal categories is owed in part to the fact that they present composition—tensoring for a monoidal category considered as composition in the one-object bicategory—in a completely unbiased fashion.

tively too. To that end, because $f$ and $f'$ are parallel let us fix the values

$$\alpha \xleftarrow{\Theta} \alpha' \quad \xleftarrow{\frac{\kappa(f,f')}{f \qquad f'}}\!\!\dashv \quad \{\pi\} \qquad \text{and} \qquad \alpha' \xleftarrow{\Psi} \alpha \quad \overset{\frac{\kappa(f',f)}{f' \qquad f}}{\Longleftarrow\!=\!=} \quad \{\pi\}$$

given by the split contraction $\kappa$ on $\sigma$. These will serve as our mediating cells (Definition 3.2.4 (i)) and so let us determine also evaluations (Definition 3.2.4 (ii))

$$\alpha \xleftarrow{\Theta\Psi} \alpha \quad \overset{\frac{\kappa(\alpha)}{\alpha \qquad \alpha}}{\Longleftarrow\!=} \quad \alpha \xleftarrow{\Theta} \alpha', \xleftarrow{\Psi} \alpha \qquad \text{and} \qquad \alpha' \xleftarrow{\Psi\Theta} \alpha \quad \overset{\frac{\kappa(\alpha')}{\alpha' \qquad \alpha'}}{\Longleftarrow\!=\!=} \quad \alpha', \xleftarrow{\Psi} \alpha \xleftarrow{\Theta} \alpha'$$

in $\mathbb{A}$ from the split contraction $\kappa$ on $d^{\mathbb{A}}$. Observe that $\kappa(f,f')$ and $\kappa(f',f)$ are pasteable proofs, and liable for the action of the codomain with $\kappa(\alpha)$ and $\kappa(\alpha')$. From this we obtain proofs that the evaluations $\Theta\Psi$ and $\Psi\Theta$ are both values of $F$ on $\{\pi\}$.

$$\alpha \xleftarrow{\Theta\Psi} \alpha \quad \xleftarrow{\frac{\kappa(\alpha) \,\square^{\tau}\, (\kappa(f,f')|\kappa(f',f))}{f \qquad\qquad\qquad\qquad f}}\!\!\dashv \quad \{\pi\} \quad \text{and} \quad \alpha' \xleftarrow{\Psi\Theta} \alpha' \quad \xleftarrow{\frac{\kappa(\alpha') \,\square^{\tau}\, (\kappa(f',f)|\kappa(f,f'))}{f' \qquad\qquad\qquad\qquad f'}}\!\!\dashv \quad \{\pi\}$$

Now let us appeal to Lemma 3.2.3 to obtain identities (Definition 3.2.4 (iii))

$$\alpha \xleftarrow{1_\alpha} \alpha \quad \overset{\frac{r}{\alpha\ \alpha}}{\Longleftarrow} \quad \{\alpha\} \qquad \text{and} \qquad \alpha' \xleftarrow{1_{\alpha'}} \alpha' \quad \overset{\frac{s}{\alpha'\ \alpha'}}{\Longleftarrow} \quad \{\alpha'\}\ .$$

To give the remaining equivalences demanded, $1_\alpha \sim \Theta\Psi$ and $\Psi\Theta \sim 1_{\alpha'}$, we wish to coinductively apply the proof thus far. However, we have not yet realised the reference identities as values of $F$.

For this purpose let us observe that we may utilise the action of the domain $\mathbb{B}$ on $F$ to obtain proofs

$$\alpha \xleftarrow{1_\alpha} \alpha \quad \xleftarrow{\frac{r \,\square^{\sigma}\, \{f\}}{f \qquad f}}\!\!\dashv \quad \{\pi\} \qquad \text{and} \qquad \alpha' \xleftarrow{1_{\alpha'}} \alpha' \quad \xleftarrow{\frac{s \,\square^{\sigma}\, \{f'\}}{f' \qquad f'}}\!\!\dashv \quad \{\pi\}$$

so that the proof is complete by coinduction. ■

Next let us make good on our promise that proof-relevant functors take equivalences to equivalences.

**Lemma 4.1.4.** *Let $F\colon \mathbb{B} \to \mathbb{A}$ be a proof-relevant functor and suppose there is the data of an equivalence $x \sim y$ in $\mathbb{B}$. For any pair of proofs*

$$Fx \xleftarrow{f_x}\!\!\dashv \ x \qquad \textit{and} \qquad Fy \xleftarrow{f_y}\!\!\dashv \ y$$

*there is the data of an equivalence $Fx \sim Fy$.*

*Proof.* As demanded by the coinductive nature of the definition of equivalence, our overall strategy will be coinductive. With that in mind, let us write

$$x \xleftarrow{\alpha\beta} x \overset{p}{\Leftarrow} x \xleftarrow{\alpha} y \xleftarrow{\beta} x \qquad \text{and} \qquad y \xleftarrow{\beta\alpha} y \overset{q}{\Leftarrow} y \xleftarrow{\beta} x \xleftarrow{\alpha} y$$

for the data of the mediating cells of $x \sim y$ and the evaluations of their pastings in the proof-relevant category $\mathbb{B}$. To begin let us fix the mediating cells of the to-be equivalence $Fx \sim Fy$ using the split contraction $\kappa$ on $\sigma$.

$$Fx \xleftarrow{F\alpha} Fy \overset{\frac{\kappa(\alpha)}{f_x \quad f_y}}{\longleftarrow\!\!\!\dashv} x \xleftarrow{\alpha} y \qquad \text{and} \qquad Fy \xleftarrow{F\beta} Fx \overset{\frac{\kappa(\beta)}{f_y \quad f_x}}{\longleftarrow\!\!\!\dashv} y \xleftarrow{\beta} x$$

Using the split contraction $\kappa^{\mathbb{A}}$ of $\mathbb{A}$ let us also determine evaluations for their pastings.

$$Fx \xleftarrow{F\alpha F\beta} Fx \overset{\frac{\kappa(F\alpha)}{Fx \qquad Fx}}{\Longleftarrow\!=\!=} Fx \xleftarrow{F\alpha} Fy \xleftarrow{F\beta} Fx$$

$$Fy \xleftarrow{F\beta F\alpha} Fy \overset{\frac{\kappa(F\beta)}{Fy \qquad Fy}}{\Longleftarrow\!=\!=} Fy \xleftarrow{F\beta} Fx \xleftarrow{F\alpha} Fy$$

Finally for reference identities let us appeal to Lemma 3.2.3 to obtain

$$Fx \xleftarrow{1_{Fx}} Fx \overset{\frac{r}{Fx\ Fx}}{\Longleftarrow\!=} \{Fx\} \qquad \text{and} \qquad Fy \xleftarrow{1_{Fy}} Fy \overset{\frac{s}{Fy\ Fy}}{\Longleftarrow\!=} \quad .$$

To complete the proof we must now give equivalences $1_{Fx} \sim F\alpha F\beta$ and $F\beta F\alpha \sim 1_{Fy}$. To give these, we shall instead give equivalences

$$1_{Fx} \overset{(1)}{\sim} F1_x \overset{(2)}{\sim} F(\alpha\beta) \overset{(3)}{\sim} F\alpha F\beta$$
$$F\beta F\alpha \overset{(3)}{\sim} F(\beta\alpha) \overset{(2)}{\sim} F1_y \overset{(1)}{\sim} 1_{Fy} \ ,$$

and appeal to Lemma 3.2.13 (i) for transitivity. In the above we have written $F1_x$ and $F1_y$ for intended values of $F$ on the reference identities of $x \sim y$, and $F(\alpha\beta)$ and $F(\beta\alpha)$ for intended values of $F$ on the evaluations of the pastings of the mediating cells of $x \sim y$. To obtain these we may appeal to the split contraction $\kappa$ on $\sigma$.

This notation is further intended to suggest that the equivalences marked (1) and (3) are formal consequences of the actions and laws governing proof-relevant functors, and indeed this is so. Both of these equivalences are instances of Lemma 4.1.3 above for we may suitably recast $1_{Fx}$ as a value of $F$ on $\{x\}$ with a proof parallel to that of $F1_x$, and similarly so for the others. Thus we are left to contend with the equivalences marked (2). However, these are precisely a coinductive application of the proof thus far as applied to the higher equivalence data $1_x \sim \alpha\beta$ and $\beta\alpha \sim 1_y$ and the aforementioned values of $F$. ■

Finally let us attend to the notion of presented equivalences as was discussed in

Remark 3.2.17. Observe the drastic simplification in proof and increase in strength of conclusion in contrast to the generic equivalence case above.

**Lemma 4.1.5.** *Let $F\colon \mathbb{B} \to \mathbb{A}$ be a proof-relevant functor. If there are a pair of cells $x$ and $y$ of the same dimension in $\mathbb{B}$, and a proof*

$$x \overset{p}{\Leftarrow} y$$

*then every value $F$ obtains on $x$ in $\mathbb{A}$ is also a value of $F$ on $y$. Consequently, if there is additionally a proof*

$$y \overset{q}{\Leftarrow} x$$

*then $F$ obtains precisely the same values on $x$ and $y$.*

*Proof.* This is a straightforward consequence of the action of the domain $\mathbb{B}$ on $F$. If we have a proof $f$ as below-left that $\alpha$ is a value of $F$ on $x$ then by $\square^\sigma$ we obtain a corresponding proof as below-right that $\alpha$ is a value of $F$ on $y$.

$$\alpha \overset{f}{\longleftarrow\!\!\!\!\shortmid} x \qquad \text{and} \qquad \alpha \xleftarrow{f \,\square^\sigma p}\!\!\!\!\shortmid\ y$$

Note that we claim only that the values obtain are the same on $x$ and $y$, not necessarily the proofs for these values. ■

## 4.2. Examples

Now that we have seen and understood several key aspects of proof-relevant functors, let us turn our attention to some examples.

### 4.2.1 Proof-relevant categories as identity proof-relevant functors

Before we produce any particular examples of proof-relevant functors, we would like to note that there is an especially abundant source. In comparing Definition 4.1.1 with Definition 3.1.1 we might observe that every proof-relevant category is in fact canonically a proof-relevant endofunctor. Indeed, if $\mathbb{A} = A \leftarrow P \to TA$ is a proof-relevant category, should we set $F = P$ and both of the actions to proof composition then the laws are satisfied and there is a split contraction on the domain leg. Let us record this observation here.

**Lemma 4.2.1.** *Each proof-relevant category $\mathbb{A}$ is canonically endowed with the structure of a proof-relevant functor $\mathbb{A}\colon \mathbb{A} \to \mathbb{A}$.* ■

*Remark* 4.2.2. The observation that each proof-relevant category is a proof-relevant endofunctor explains the close parallels in the proofs of Lemmas 4.1.3 and 3.2.7—we now understand the latter to be a special case of the former! In fact, on this view Lemma 4.1.4 gives a heretofore un-noticed result when specialised to proof-relevant categories: given $x \sim y$, any two evaluations of $x$ and $y$ are themselves equivalent. ◂

What should Lemma 4.2.1 mean? Let us suggest here that a proof-relevant category viewed instead as a proof-relevant functor is *an* identity in some putative proof-relevant category of proof-relevant categories. Although we do not make any significant inroads to constructing such a device in this work, we nevertheless see several hints of identity-like behaviour.

The first is at the most literal level: understood as a proof-relevant functor, a proof-relevant category dictates that the possible values that a pasting obtains are precisely its possible evaluations—nothing is changed. Next, continuing the analogy with bimodules mentioned in Remark 4.1.2, we understand that what we have observed is tantamount to the idea that any category gives rise to its "hom bimodule" which certainly has identity-like properties among other bimodules.[7] Finally, in Section 5.5 we put forth a tentative notion of composition for proof-relevant functors and both directly and through the apparatus of the enclosing section we are able to prove that proof-relevant categories behave as units for composition.

It therefore would appear that the words of [EM45] ring true for us: the whole concept of a proof-relevant category seems auxiliary to that of proof-relevant functor.

### 4.2.2 Strict functors

Proof-relevant functors genuinely generalise the notion of strict functor, as we shall now see. Recall that in Section 3.3.1 we saw how each strict algebra of $T$, that is each $\omega$-category, is canonically a proof-relevant category. Between these especially simple proof-relevant categories we may recover the $\omega$-functors as precisely those proof-relevant functors which are single-valued, in the following sense.

**Lemma 4.2.3.** *Let $(A, h^A)$ and $(B, h^B)$ be $\omega$-categories viewed as proof-relevant categories $\mathbb{A}$ and $\mathbb{B}$. There is a bijection between $\omega$-functors $(B, h^B) \to (A, h^A)$ and proof-relevant functors $F\colon \mathbb{B} \to \mathbb{A}$ such that $\sigma\colon F \to TB$ is an isomorphism of globular sets.*

*Proof.* This proof reduces to the following, less-publicised characterisation of morphisms of algebras. There is a bijection of sets

$$\left\{ \begin{array}{c} \text{maps } A \xleftarrow{F} TB \text{ satisfying} \\ F(Th^B) = F\mu_B = h^A(TF) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{algbera morphisms} \\ (A, h^A) \xleftarrow{f} (B, h^B) \end{array} \right\},$$

which is implemented by the assignments $F \mapsto F\eta_B$ and $f \mapsto fh^B = h^A(Tf)$. Finally, proof-relevant functors $F\colon \mathbb{B} \to \mathbb{A}$ whose leg $\sigma\colon F \to TB$ is an isomorphism may be seen to be in bijection with the first set, for the split contraction structure is uniquely determined. ∎

This lemma also witnesses an important degeneration of attributes for proof-relevant functors. Not only are proof-relevant functors as in the hypothesis of the

[7] Quite what "hom bimodule" is intended to mean in the asymmetric world of $T$-categories we do not pretend to understand.

lemma single-valued, but as we see from the proof they arise from algebra morphisms by one of two equal ways, $fh^B = h^A(Tf)$. That is, such proof-relevant functors have values that may be simultaneously understood as the result of compose-then-map $(fh^B)$ and map-then-compose $(h^A(Tf))$.

### 4.2.3 Generalised anafunctors

Let us now forgo the constraint of single-valued-ness of the proceeding example, and examine the resulting nature of proof-relevant functors still nevertheless between $\omega$-categories. In order to understand these, we invite the reader to recall the notion of anafunctor as given in [Mak96, Def. 1].

Upon examination we might notice that anafunctors $\mathcal{C} \to \mathcal{E}$ of 1-categories may also be described as spans $\mathcal{C} \leftarrow \mathcal{D} \to \mathcal{E}$ of 1-categories such that the leftward functor is equipped with the data of a split contraction in the finite-dimensional sense: there is a section on 0-cells, and given any pair of 0-cells of $\mathcal{D}$ and a 1-cell of $\mathcal{C}$ over their image, there is a ***unique*** lift of the 1-cell to $\mathcal{D}$. Motivated by this, let us make the following definition.

**Definition 4.2.4.** Let $(A, h^A)$ and $(B, h^B)$ be $\omega$-categories, a **generalised anafunctor** $F\colon \mathcal{B} \to \mathcal{A}$ is a span of $\omega$-functors

$$(B, h^B) \xleftarrow{\sigma} (F, h^F) \xrightarrow{\tau} (A, h^A)$$

such that the globular morphism $\sigma\colon F \to B$ is equipped with the data of a split contraction. ◁

Let us assure ourselves that this is a genuine generalisation of the notion of anafunctor.

**Lemma 4.2.5.** *Let $\mathcal{B}$ and $\mathcal{A}$ be 1-categories discretely embedded into $\omega$-categories. Then a generalised anafunctor $F\colon \mathcal{B} \to \mathcal{A}$ is an anafunctor.*

*Proof.* This claim hinges on the assertion that there are unique 1-cells in $F$ above 1-cells of $\mathcal{B}$. That is, that a split contraction on $\sigma$ in this case really does have the uniqueness property we desire.

Suppose then that there are parallel 1-cells $\varphi, \varphi'$ in $F$ on the same 0-cells $f, f'$ such that $\sigma\varphi = \sigma\varphi'$, then by the split contraction applied to the 2-cell $* = \mathrm{id}_{\sigma\varphi}$ of $\mathcal{B}$ there must be a 2-cell between $\varphi$ and $\varphi'$, but therefore $\varphi = \varphi'$. ■

*Remark* 4.2.6. Much like the original anafunctors of Makkai, generalised anafunctors may also be composed suitably weakly. The composition proceeds by here by taking pullbacks, and the reader is invited to check that the stability of split contractions under pullback and composition (Lemmas 2.4.5 (ii) and 2.4.5 (iii)) is all that is required to complete this process. ◁

Finally let us see that all generalised anafunctors, and therefore anafunctors, give examples of proof-relevant functors. The idea is that we may extend an anafunctor to obtain values on pastings by first composing the pastings strictly in the domain and then observing the possible values of the composite.

**Lemma 4.2.7.** *Let $F\colon \mathcal{B} \to \mathcal{A}$ be a generalised anafunctor between $\omega$-categories. Then the span $A \leftarrow F^a \to TB$ defined by the pullback*

$$\begin{tikzcd}
F^a \arrow[r] \arrow[d, "\sigma'"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[r, "\tau"] \arrow[d, "\sigma"] & A \\
TB \arrow[r, "h^B"'] & B &
\end{tikzcd}$$

*is canonically equipped with the structure of a proof-relevant functor.*

*Proof.* First let us note that $\sigma'$ is endowed with the data of a split contraction by Lemma 2.4.5 (iii), for it is the pullback of $\sigma$. To give the actions of the domain and codomain on $F^a$ we must compute some pullbacks, but inevitably these pullbacks involve applying $T$ to $F$ and $B$ so that the actions are defined in terms of the algebra structures $(F, h^F)$, $(B, h^B)$, and the monad multiplication. ■

To conclude this section, we wish to point out that while it is tempting to imagine that all proof-relevant functors between $\omega$-categories must be of the form $F^a$ above for some generalised anafunctor this is in fact not the case and we will explore such an example in Section 4.2.5.

### 4.2.4 Classical pseduo-functors

In the previous sections we were interested in examples which progressed to proof-relevant functors from $\omega$-functors by relaxing the idea of multi-valued-ness but otherwise were closely coupled to the world of strict devices. In this section we will sketch another way in which we might obtain examples of proof-relevant functors: by capturing extant, weak functors between higher categories.

In particular we have in mind that every pseudo-functor of bicategories should give an example of a proof-relevant functor when the two bicategories concerned are viewed as proof-relevant categories as in Section 3.3.3. It turns out that this is rather involved, although most of the complexity is inessential to the core idea. For that reason we provide only a sketch (or perhaps, a suggestions) of the construction.

Suppose then that we have two classical bicategories $\mathcal{K}$ and $\mathcal{L}$, and a pseudo-functor $F\colon \mathcal{L} \to \mathcal{K}$. Let us write $\mathbb{P}$ for the contractible operad whose algebras are classical bicategories. With this, we desire to realise $F$ as a span $\widehat{F}$

$$\begin{tikzcd}
K & T_{\mathbb{P}}K \arrow[l] \arrow[r] & TK \\
& \widehat{F} \arrow[ul, "\sigma"] \arrow[dr, "\tau"] & \\
L & T_{\mathbb{P}}L \arrow[l] \arrow[r] & TL
\end{tikzcd},$$

along with the structure of a split contraction on $\sigma$ and actions of the domain and codomain. In dimension zero we may simplify our work by setting $(\widehat{F})_0 :\equiv L_0 = (TL)_0$

so that $\widehat{F}$ obtains only a single value on each object. Given that $\mathbb{P}$ has only the trivial operation in dimension zero there is nothing more to be done at this level.

In dimension one, we wish to set the 1-cells $\varphi \in (\widehat{F})_1$ over a pasting $\pi \in (TL)_1$ to be "generated" by the pairs

$$(\text{a composite } f \text{ of } \pi \text{ in } \mathcal{L}, Ff) \qquad \text{and} \qquad (\text{a composite } g \text{ of } (TF)\pi \text{ in } \mathcal{K}, g)$$

so that we may take $\tau$ and $\sigma$ to be the evident projections. First, by "a composite" we mean here the full data of a proof of composition in a classical bicategory: a choice of evaluation strategy. By "generated" we mean to take also pairs that involve some arbitrary combination of these two procedures, say mapping some part of the pasting over after composition in $\mathcal{L}$ and performing the remaining composition in $\mathcal{K}$. In this way we account for the actions of the domain and the codomain which perform essentially such decompositions. Unfortunately, it would appear that great care and effort must be expended to make this particular idea rigorous and for that the overall construction suffers. Let us therefore conclude our sketch and direct any future endeavours to unearth the many details we omitted by suggesting that the 2-cells of $\widehat{F}$ are uniquely assigned between 1-cells in accordance with the unitors and compositors of $F$, so that we may witness the expected coherences of a pseudo-functor.

### 4.2.5 Proof-relevant functors defined by universal constructions

The last example to which we wish to draw the reader's attention is the case of proof-relevant functors naturally emerging from universal constructions. We hope to convince the reader that the nature of proof-relevant functors—multi-valued-ness, and obtaining values on a pasting instead of only on cells—are inevitable consequences of taking seriously the idea that no one value of a universal construction should be privileged over any other, and that such constructions are often applicable to more than one object at a time.

Perhaps the most concrete place to perform this exercise is in the category of sets in the context of taking products and coproducts. Let us therefore call by $\mathbb{S}$et the proof-relevant category obtained in the spirit of $\mathbb{F}$DVect$_k$ of Section 3.3.4: a proof-relevant version of a cocartesian monoidal structure on Set. The proof-relevant category $\mathbb{S}$et therefore has a unique 0-cell *, each set as a 1-cell, each, each function as a 2-cell, and only discrete cells higher up.

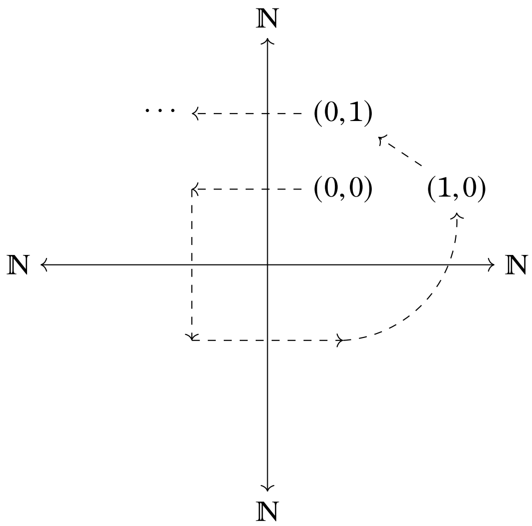
There is only a single proof $* \Leftarrow *$ in dimension zero, and in dimension one a proof that a set $C$ is an evaluation of a pasting, that is a finite list of sets $\langle A_i \rangle$, is a cocone $(C, \iota)$ on the discrete diagram of the $A_i$ and a proof that this cocone is colimiting. That is, $C$ should be proven to be *a* coproduct. Finally, the proofs in dimension two are discretely added between any two cocones over the same diagram.

On this proof-relevant category, the fully unbiased version of $(\mathrm{Set}, +, \emptyset)$ which is attentive to dependence of sums on their summands, we wish to describe an endomorphism which is determined by "multiplying by a set $S$". Let us suggestively call such a putative assignment $S \times (-)$.

Immediately we see that it would be cruel to demand choosing but one value of a product $S \times A$ for a set $A$, and so this assignment ought to be multi-valued. In fact, however, it is not only the values on a fixed set which may be multiplicitous: there may be in general distinct reasons *why* even a single value $X$ ought to be a product of $S \times A$. For instance, the encoding of $\mathbb{N}$ as a product $\mathbb{N} \times \mathbb{N}$ may proceed by at least two distinct ways by snaking through the table $\mathbb{N} \times \mathbb{N}$ in patterns which are symmetric about the diagonal. Thus we see that by taking seriously the idea of recording the relevant details of a universal construction we are forced into having potentially many proofs and values for a fixed cell in the domain.

More still is true, however. Given a finite list of sets $\langle A_i \rangle$, despite the fact that products by a fixed set distribute over finite coproducts, there are nevertheless particular reasons that $S \times (A+B)$ is a product which are not literally obtained from reasons that $S \times A, S \times B$ are products or reasons that $A + B$ is a coproduct.

To see an example of this, consider $S = \mathbb{N}+\mathbb{N}$ and $A = B = \mathbb{N}$. There is a particular proof that $\mathbb{N}$ is a product $\mathbb{N} \times (\mathbb{N} + \mathbb{N})$ which draws on all of the information at hand simultaneously: consider $(\mathbb{N} + \mathbb{N}) \times (\mathbb{N} + \mathbb{N})$ as four tables arranged as follows




then we may give an enumeration of these numbers by spiralling outwards from the centre, as indicated. This proof does not be decompose into proofs involving first computing $(\mathbb{N} + \mathbb{N}) \times \mathbb{N}$ twice or first computing $\mathbb{N} + \mathbb{N}$.

This excursus may be summarised as follows: by demanding that the only proofs that a value is obtained on a pasting originate from map-then-compose or compose-then-map decompositions, we lose information.

# 5. Proof-relevant algebras

In his PhD thesis [Lei04, §4], Leinster writes

> *There is also a notion of what a contraction on a map of $T$-multicategories is. ...[It] has the property that for $T$-operads $\mathbb{C}$, a contraction on the unique map from $\mathbb{C}$ to the terminal $T$-operad is precisely a contraction on $\mathbb{C}$ ...*

but gives no formal definition. It is the goal of this section to extend this suggestion to a general framework for specifying not *strict* algebras for a given $T$-category, but rather *proof-relevant algebras*. By leveraging this framework, as well as the shape functor of Section 5.4 below, we will be able to give the definition of *proof-relevant* versions of all devices of category theory specified by (potentially) higher morphisms and equations on their composites: categories, functors, functor composition, natural transformations, monads, and so on.

To work towards this we organise the material as follows. In Section 5.1 we develop a definition of proof-relevant algebra for a $T$-category in analogy with the definition of algebra. Then in Section 5.2 we prove an important early result: proof-relevant algebras are monadic in the expected fashion. In Section 5.3 we compare various features of the theory of algebras with that of proof-relevant algebras and encounter devices for yielding further examples of proof-relevant algebras. After this, in Section 5.4, we give one of the central definitions of this work: the shape functor from $\omega$-categories to $T$-categories. This functor allows us to essentially design proof-relevant algebras with desired features. Finally in Section 5.5 we look at a special case of the shape functor and explore a notion of proof-relevant functor composition.

## 5.1. The definition

Our goal in this section is to work towards defining proof-relevant algebras for a $T$-category $\mathbb{C}$ in a manner analogous to (strict) algebras, but with one important difference. In the case of algebras, $\mathbb{C}$ dictates genuine operations on arrangements of cells of the algebra. We wish instead that a proof-relevant algebra supply a coherent system of proofs that the operations of $\mathbb{C}$ paired with arrangements of cells of the algebra evaluate to other cells.

Let us retrace therefore the construction of algebras for a $T$-category $\mathbb{C}$. Recall Theorem 2.2.11, the category of algebras for $\mathbb{C}$ is equivalent to the category of discrete opfibrations over $\mathbb{C}$. Should we examine Definition 2.2.10 to recall the meaning of the latter term, we might note that a discrete opfibration is a combination of data and a property.

The data of a discrete opfibration is a morphism $F\colon \mathbb{D} \to \mathbb{C}$ of $T$-categories, and the property is that a certain square is a pullback. However, this property may be reformulated at the level of $T$-graphs, for it is independent of the structure of the

$T$-categories at hands. Thus let us begin by identifying one of the key components of discrete opfibrations *qua* algebras.

**Definition 5.1.1.** If $C$ is a $T$-graph, then there is a functor $P\colon T\text{-Grph}/C \to \mathrm{Ar}\,\widehat{\mathbb{G}}$ defined on objects $f\colon A \to C$ as follows

$$\begin{tikzcd}
A^1 \arrow[drr, bend left, "A^r"] \arrow[dr, dashed, "Pf"] \arrow[ddr, bend right, "f^1"'] & & \\
& C^1 \times_{TC^0} TA^0 \arrow[r, "\pi_A"] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & TA^0 \arrow[d, "Tf^0"] \\
& C^1 \arrow[r, "C^r"'] & TC^0
\end{tikzcd} , \tag{5.1.2}$$

and which is extended to morphisms by the universal property of pullbacks. ◁

From this we see that we may give an alternate description of discrete opfibrations, one more amenable to generalisation in the direction we intend. By inspection, we may deduce the following.

**Lemma 5.1.3.** *The category* $\mathrm{Alg}\,\mathbb{C}$ *fits into the following pullback in* Cat.

$$\begin{tikzcd}
\mathrm{Alg}\,\mathbb{C} \arrow[r, "\text{forget}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-Cat}/\mathbb{C} \arrow[d, "U'"] \\
(T\text{-Grph}/U\mathbb{C})^{\mathrm{DOF}} \arrow[r, "\text{forget}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-Grph}/U\mathbb{C} \arrow[d, "P"] \\
\mathrm{Iso}\,\widehat{\mathbb{G}} \arrow[r, "\text{forget}"'] & \mathrm{Ar}\,\widehat{\mathbb{G}}
\end{tikzcd}$$

*In the above, the category* $\mathrm{Iso}\,\widehat{\mathbb{G}}$ *is the full sub-category of* $\mathrm{Ar}\,\widehat{\mathbb{G}}$ *whose objects are isomorphisms, and the functor* $U'$ *is the evident forgetful functor induced by* $U\colon T\text{-Cat} \to T\text{-Grph}$. *Additionally, the category* $(T\text{-Grph}/U\mathbb{C})^{\mathrm{DOF}}$ *is defined by its pullback.* ■

The crucial observation of this lemma is that we may decompose the construction of algebras into two general stages depending essentially on a single choice: we need only identify the property (or more generally structure) we wish to have on graphs by selecting the bottom-left category in the above pullbacks, and the rest follows.

Let us spend a moment to better understand how this lemma presents a decomposition of the classical notion of algebra. With reference to (5.1.2), a discrete opfibration of $T$-graphs in the image of $U'$, that is an object of $(T\text{-Grph}/U\mathbb{C})^{\mathrm{DOF}}$, would have an isomorphism $Pf\colon A^1 \xrightarrow{\cong} C^1 \times_{TC^0} TA^0$. Recall that classically, in the context of algebras, the globular set $C^1$ may be though of as the globular set of operations of the $T$-category $\mathbb{C}$. These operations are predicated on pastings of formal sorts, those elements of $TC^0$, and produce a target sort.

Thus we should think of $C^1 \times_{TC^0} TA^0$ as the globular set of pairs of operations of $\mathbb{C}$ with matching pastings of cells of $\mathbb{A}$ fibred correctly over the formal pasting of sorts the operation expects. Inasmuch as a discrete opfibration of $T$-graphs requires $Pf$ to be invertible, it declares that the operations of $\mathbb{A}$ are precisely just what we have described: operations of $\mathbb{C}$ whose arguments are filled by correctly sorted cells of $\mathbb{A}$.

There is another crucial aspect to this story. Algebras are not free on the operations of $\mathbb{C}$, and equalities between formal composites of formal operations of $\mathbb{C}$ must be obeyed by each algebra at the level of actions. The way that this is encoded is through the morphism $f\colon \mathbb{A} \to \mathbb{C}$ of $T$-categories. Should we perform some composition of operations in $\mathbb{A}$, the result would have to be fibred over the composition of the underlying constituent formal operations in $\mathbb{C}$, precisely because $F$ is a morphism. But in fact more is true: $Pf$ is an isomorphism, so the result of composing operations in $\mathbb{A}$ must in fact be precisely the pair of the composite of the formal operations in $\mathbb{C}$ and the matching overall pasting of sorted cells of $\mathbb{A}$. Thus, any equation which holds in $\mathbb{C}$ for composites of formal operations holds too in $\mathbb{A}$ for such pairs.

With this understanding in mind, the path to proof-relevant algebras almost suggests itself. Should we replace $\mathrm{Iso}\,\widehat{\mathbb{G}}$ with the category $\mathrm{SCon}\,\widehat{\mathbb{G}}$ of split contractions, we would thereby derive a notion of proof-relevant fibration of $T$-graphs. In such a fibration the map $Pf$ would not be required to be an isomorphism, but rather equipped with the data of a split contraction. In the vocabulary of the above discussion, this would have the following implications for $\mathbb{A}$.

Given any pair of operation of $\mathbb{C}$ and arguments filled by appropriately sorted cells of $\mathbb{A}$, that $Pf$ is not invertible means that there would be in general many operations—hereafter proofs—in $A^1$ in the fibre above this pair. These proofs thus bear the interpretation of proving that such a pair could be evaluated to a correctly sorted result in $\mathbb{A}$, consequently in general there would be many evaluations. Owing to the split contraction, there would always be at least one such proof (Lemma 2.4.3) and so every operation of $\mathbb{C}$ could be evaluated on correctly sorted pastings of cells of $\mathbb{A}$. Furthermore, given any operation of $\mathbb{C}$ and correctly sorted cells of $\mathbb{A}$, any two parallel proofs that evaluate the two boundaries of this operation have above them at least one proof that this operation itself may be evaluated over our chosen boundary evaluations.

This is moving, but there is a crucial aspect as yet unaddressed: what becomes of the equations between composites of formal operations in $\mathbb{C}$? It is still the case that a composite of proofs in $\mathbb{A}$ must be fibred over the composite of the underlying formal operations of $\mathbb{C}$. However $Pf$ is no longer an isomorphism and so there is no sense in which the equations between formal operations in $\mathbb{C}$ hold strictly. However, there is a sense in which these equations hold up to equivalence. Consider two proofs $p$ and $q$ in the same fibre over $C^1$, paired as they are with the same formal operation $\Psi$ of $\mathbb{C}$ as well as the *same* pasting $\pi$ of cells of $\mathbb{A}$ sorted as it must be to match the arguments of $c$. This situation captures the most general setup for the following supposition: assume additionally that $p$ is a composite of other proofs $p_i$ over other formal operations $\Psi_i$, such that the composite $\mathsf{comp}(\Psi_i)$ of the formal operations is $\Psi$. That is, we have now placed ourselves in the context of considering the effect of an equality of formal operations in $\mathbb{C}$.

If $p$ and $q$ are parallel, then the split contraction on $Pf$ assures us of the data of an equivalence (in the sense of Definition 3.2.4) between their two evaluations, contained entirely in the fibre above the sort of the target of $\Psi$—seen by essentially the argument of Lemma 3.2.7. If $p$ and $q$ are not parallel, then nevertheless we suggest that beginning with their zero-dimensional components we may generate equivalences under whose conjugation the two evaluations are eventually found to be equivalent. In this way, the equations of $\mathbb{C}$ lead to *equivalences* between evaluations in $\mathbb{A}$.

We hope that what we have described to the reader is a reasonably evocative notion of proof-relevancy for algebras, and so we will press on with the definitions to realise this rigorously.

**Definition 5.1.4.** The **category of proof-relevant fibrations of $T$-graphs over $C$**, written $(T\text{-Grph}/C)^{\mathrm{PRF}}$, is defined to be the vertex of the following pullback in Cat.

$$\begin{tikzcd}
(T\text{-Grph}/C)^{\mathrm{PRF}} \arrow[r, "V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-Grph}/C \arrow[d, "P"] \\
\mathrm{SCon}\,\widehat{\mathbb{G}} \arrow[r] & \mathrm{Ar}\,\widehat{\mathbb{G}}
\end{tikzcd}$$

Thus a proof-relevant fibration of $T$-graphs over $C$ is a morphism $f\colon B \to C$ of $T$-graphs together with the data of a split contraction on $Pf$ of Definition 5.1.1. ◁

From this category of proof-relevant fibrations of $T$-graphs, we continue in the fashion laid out in Lemma 5.1.3 above and arrive at our definition of proof-relevant algebra.

**Definition 5.1.5.** The category of **proof-relevant $\mathbb{C}$-algebras** for a $T$-category $\mathbb{C}$ is defined to be the vertex of the following pullback in Cat.

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{PR}}\mathbb{C} \arrow[r, "V^{\mathrm{PR}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-Cat}/\mathbb{C} \arrow[d, "U'"] \\
(T\text{-Grph}/U\mathbb{C})^{\mathrm{PRF}} \arrow[r, "V"'] & T\text{-Grph}/U\mathbb{C}
\end{tikzcd}$$

Thus a proof-relevant $\mathbb{C}$-algebra is a morphism $F\colon \mathbb{B} \to \mathbb{C}$ of $T$-categories together with the data of a split contraction on the canonical map $PF$. ◁

*Remark* 5.1.6. Note that we have defined a category of proof-relevant algebras, but that the morphisms which inhabit it are strict in every possible way. The matter of suitably weak morphisms between proof-relevant algebras is somewhat more delicate, and we provide some suggestions in Section 5.4. ◁

**Example 5.1.7**

The terminal $T$-category 1 is given by the unique $T$-category structure on the terminal $T$-graph $1 \leftarrow T1 = T1$. From this description we see that a proof-relevant algebra for 1 is any $T$-category $\mathbb{C}$ together with the data of a split contraction on the canonical map $P!$ in (5.1.2). Should we compute this pullback, we see that 1 sufficiently degenerates matters to leave us with the following characterisation: a proof-relevant algebra for 1 is a $T$-category $\mathbb{C}$ together with the data of a split contraction on its domain leg $C^1 \to TC^0$. Said another way, a proof-relevant algebra for 1 is a proof-relevant category.

**Example 5.1.8**

There is a certain $T$-category $\mathbb{S}2$ whose data is given by the $T$-graph we schematically illustrate below.

$$1 + 1 \xleftarrow{c} T1 + T1 + T1 \xrightarrow{d} T(1 + 1)$$

Its comp and refl are essentially dictated by $\mu$ and $\eta$, and for now we do not elaborate on this further—a rigorous definition shall be obtained in Section 5.4.

Nevertheless, let us compute what a $T$-category fibred over this amounts to. Because $\widehat{\mathbb{G}}$ is infinitary extensive we may rewrite $A^0 \to 1 + 1$ as $C^0 + D^0 \to 1 + 1$ by taking pullbacks. Continuing in this way to decompose $\mathbb{A} \to \mathbb{S}2$ we find that such a $T$-category $\mathbb{A}$ is in fact the data of two $T$-categories $\mathbb{C}$ and $\mathbb{D}$, as well as the data of a $(\mathbb{C}, \mathbb{D})$-bimodule which we shall suggestively call $F$. Furthermore, the pullback we must compute in (5.1.2) decomposes as the coproduct of three pullbacks because $T$ is coproduct preserving and $\widehat{\mathbb{G}}$ is extensive. In order to render $\mathbb{A}$ a proof-relevant algebra, we must therefore supply the data of three split contractions.

All in all, a proof-relevant algebra for $\mathbb{S}2$ is the data of two $T$-categories $\mathbb{C}$, $\mathbb{D}$, as well as the data of a $(\mathbb{C}, \mathbb{D})$-bimodule $F$, and split contractions on the legs $C^1 \to TC^0$, $D^1 \to TD^0$, and $F \to TD^0$. That is, a proof-relevant algebra for $\mathbb{S}2$ is precisely a pair of proof-relevant categories $\mathbb{C}$ and $\mathbb{D}$, and a proof-relevant functor $F : \mathbb{D} \to \mathbb{C}$.

## 5.2. Proof-relevant algebras are finitarily monadic

Now that we produced motivation for, given the definition of, and seen some examples of proof-relevant algebras, let us turn to consider an important early aspect of their theory. This section is dedicated to establishing that proof-relevant algebras are monadic over several categories of interest.

**Theorem 5.2.1.** *The forgetful functor* $\mathrm{Alg}^{\mathrm{PR}}\mathbb{C} \to T\text{-}\mathrm{Grph}/\mathbb{C}$ *is finitary and monadic, and moreover is the diagonal of the following pullback square of locally finitely presentable categories and finitary and monadic functors.*

$$\begin{tikzcd} \mathrm{Alg}^{\mathrm{PR}}\mathbb{C} \arrow[r, "V^{PR}"] \arrow[d, "U^{PR}"'] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-}\mathrm{Cat}/\mathbb{C} \arrow[d, "U"] \\ (T\text{-}\mathrm{Grph}/U\mathbb{C})^{\mathrm{PRF}} \arrow[r, "V"'] & T\text{-}\mathrm{Grph}/U\mathbb{C} \end{tikzcd} \tag{5.2.2}$$

As we wish to appeal to Theorem 2.1.5 to establish this result, we need first to prove the following theorem.

**Theorem 5.2.3.** *The functor* $V\colon (T\text{-}\mathrm{Grph}/C)^{\mathrm{PRF}} \to T\text{-}\mathrm{Grph}/C$ *is monadic and finitary for every* $T$*-graph* $C$*.*

*Remark* 5.2.4. The functor $P$ does not have a left adjoint, for it fails to preserve limits. The terminal $T$-graph fibred over $C$ is of course $C$ itself, but $PC$ is $C^1 = C^1$ which is far from the terminal object $1 = 1$ of $\mathrm{Ar}\,\widehat{\mathbb{G}}$. Regrettably this failure prohibits us from applying more sophisticated tools such as Theorem 2.1.5 to conclude that $(T\text{-}\mathrm{Grph}/C)^{\mathrm{PRF}} \to T\text{-}\mathrm{Grph}/C$ is finitary and monadic. Instead, we must carefully and elaborately obtain this result by hand and the proof of Theorem 5.2.3 is therefore deferred to the appendices. ◁

*Proof* (Theorem 5.2.3). See Appendix A. ∎

In addition, in order to prove Theorem 5.2.1 we will need the following lemmas.

**Lemma 5.2.5.** *The forgetful functor* $T\text{-}\mathrm{Cat}/\mathbb{C} \to T\text{-}\mathrm{Grph}/U\mathbb{C}$ *is finitary and monadic.*

*Proof.* By Theorems 2.5.1 and 2.2.8, the free strict $\omega$-category monad is "suitable" and so is $\widehat{\mathbb{G}}$. Thus by Theorem 2.2.8 the forgetful functor $U\colon T\text{-}\mathrm{Cat} \to T\text{-}\mathrm{Grph}$ is finitary and monadic. An application of the monadic adjunction lifting of Lemma 2.1.3 gives the desired result. ∎

**Lemma 5.2.6.** *If* $C$ *is a* $T$*-graph, then the category* $T\text{-}\mathrm{Grph}/C$ *is locally finitely presentable.*

*Proof.* To establish this fact it would be enough by, e.g. [AR94, Prop. 1.57], to show that $T$-Grph itself is locally finitely presentable. In fact, however, we can obtain a stronger result.

Let us set $F\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ to be the functor $F(X) :\equiv X \times TX$. Observe that we have an isomorphism of categories $T\text{-}\mathrm{Grph} \cong (\widehat{\mathbb{G}} \downarrow F)$ between $T$-graphs and the Artin glueing of $F$. As $\widehat{\mathbb{G}}$ is a presheaf category, the result of [CJ95, Cor. 4.4(v)] shows that $(\widehat{\mathbb{G}} \downarrow F)$ is a presheaf category if (and only if) the functor $F$ preserves wide pullbacks—but this true by Theorem 2.5.1. Thus $T\text{-}\mathrm{Grph}/C$ is a slice of a presheaf category, and so a presheaf category itself. In particular then, this category is locally finitely presentable (see for example [Bor94b, Example 5.2.2(b)]). ∎

Finally we may give the proof of Theorem 5.2.1.

*Proof* (Theorem 5.2.1). Consider that $\mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$ is the apex of the pullback (5.2.2) taken in Cat. Thus by Theorem 2.1.5, if $T$-Grph$/U\mathbb{C}$ is locally finitely presentable, and $V$ and $U$ are both finitary and monadic, then all functors (including the diagonal) are finitary and monadic and all categories are locally finitely presentable. In order, these requirements are discharged by Lemmas 5.2.5 and 5.2.6 and Theorem 5.2.3, whence the result. ∎

## 5.3. Comparisons with algebras

The next topic of study in the theory of proof-relevant algebras is their various comparisons with algebras. From these general considerations we will be able to generate an additional stock of examples.

First note that every algebra is canonically a proof-relevant algebra.

**Lemma 5.3.1.** *Let $\mathbb{C}$ be a $T$-category, then there is a fully-faithful injective-on-objects functor $I_{\mathbb{C}}\colon \mathrm{Alg}\,\mathbb{C} \to \mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$, commuting with the forgetful functors to $T$-*Cat$/\mathbb{C}$.

*Proof.* The proof amounts to the observation that for an algebra, the canonical map $PF$ of (5.1.2) is an isomorphism, and isomorphisms canonically bear the structure of split contractions.

Recall Lemma 5.1.3, so that $\mathrm{Alg}\,\mathbb{C}$ is the vertex of a pullback in Cat over the cospan $\mathrm{Iso}\,\widehat{\mathbb{G}} \to \mathrm{Ar}\,\widehat{\mathbb{G}} \leftarrow T\text{-}\mathrm{Cat}/U\mathbb{C}$. Observe that there is a fully-faithful, injective-on-objects functor $\mathrm{Iso}\,\widehat{\mathbb{G}} \to \mathrm{SCon}\,\widehat{\mathbb{G}}$ determined by Lemma 2.4.5 (i). Thus, by the universal property of pullbacks, there is fully-faithful, injective-on-objects functor $\mathrm{Alg}\,\mathbb{C} \to \mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$ whose component on $T$-Cat$/U\mathbb{C}$ is the identity. ∎

Next we may utilise the fact that split contractions are closed under composition, Lemma 2.4.5 (ii), to construct a dependent sum for proof-relevant algebras.

**Lemma 5.3.2.** *Let $\mathbb{C}$ be a $T$-category, and let $(F\colon \mathbb{B} \to \mathbb{C}) \in \mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$ be a proof-relevant $\mathbb{C}$-algebra. Then there is a faithful functor $\Sigma_F^{\mathrm{PR}}\colon \mathrm{Alg}^{\mathrm{PR}}\mathbb{B} \to \mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$ which renders commutative the diagram*

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{PR}}\mathbb{B} \arrow[r, dashed, "\Sigma_F^{\mathrm{PR}}"] \arrow[d, "V_{\mathbb{B}}^{\mathrm{PR}}"'] & \mathrm{Alg}^{\mathrm{PR}}\mathbb{C} \arrow[d, "V_{\mathbb{C}}^{\mathrm{PR}}"] \\
T\text{-}\mathrm{Cat}/\mathbb{B} \arrow[r, "\Sigma_F"'] & T\text{-}\mathrm{Cat}/\mathbb{C}
\end{tikzcd},$$

*where the vertical functors are the forgetful functors arising from the pullback definition.*

*Proof.* For the above diagram to be commutative, $\Sigma_F^{\mathrm{PR}}(G\colon \mathbb{A} \to \mathbb{B})$ must necessarily be a proof-relevant $\mathbb{C}$-algebra on the object $FG \in T$-Cat$/\mathbb{C}$. It remains then to ensure that the structure of a split contraction may be given. To that end, consider the below diagram.

$$
\begin{tikzcd}
A^1 \arrow[r, "PG"] & T_{\mathbb{B}}A^0 \arrow[r, dashed, "T_F A^0"] \arrow[d, "\pi^{\mathbb{B}}_B"'] \arrow[dr, phantom, "(3)"] & T_{\mathbb{C}}A^0 \arrow[r, "\pi^{\mathbb{C}}_A"] \arrow[d, dashed, "T_{\mathbb{C}}F^0"] \arrow[dr, phantom, "(2)"] & TA^0 \arrow[d, "TG^0"] \\
& B^1 \arrow[r, "PF"'] & T_{\mathbb{C}}B^0 \arrow[r, dashed, "\pi^{\mathbb{C}}_B"] \arrow[d, "\pi^{\mathbb{C}}_C"'] \arrow[dr, phantom, "(1)"] & TB^0 \arrow[d, "TF^0"] \\
& & C^1 \arrow[r, "d^{\mathbb{C}}"'] & TC^0
\end{tikzcd}
$$

In the above diagram, the square marked (1) is a pullback by definition. Then the pasting of the squares (1) and (2) are the defining pullback for $T_{\mathbb{C}}A^0$. As such, by pullback factorisation (2) is a pullback. Then the pasting of (3) and (2) is the defining pullback for $T_{\mathbb{B}}A^0$, so that by pullback factorisation (3) is a pullback. Finally, the map $P(FG)\colon A^1 \to T_{\mathbb{C}}A^0$ is therefore factored as the composite $(T_F A^0)PG$.

Now let us give the data of a split contraction on $P(FG) = (T_F A^0)PG$. By Lemma 2.4.5 (ii) it is enough to give the data of a split contraction on both $T_F A^0$ and $PG$, the latter of which exists by assumption. Because $T_F A^0$ is a pullback of $PF$, by Lemma 2.4.5 (iii) it suffices to give the data of a split contraction on $PF$—but this exists by assumption. To complete the proof note that $\Sigma^{\mathrm{PR}}_F$ is faithful because $\Sigma_F$ is faithful. ■

In a similar vein, because split contractions are closed under pullback we may give a re-indexing functor along any morphism of $T$-categories.

**Lemma 5.3.3.** *The assignment* $\mathbb{C} \mapsto \mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$ *may be extended to morphisms of $T$-categories via pullback to determine a pseudo-functor* $\mathrm{Alg}^{\mathrm{PR}}\colon T\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}$.

*Proof.* Given a proof-relevant algebra $F\colon \mathbb{B} \to \mathbb{C}$ and a morphism $G\colon \mathbb{D} \to \mathbb{C}$ of $T$-categories, by pullback we may compute at least another morphism $G * F\colon G * \mathbb{B} \to \mathbb{D}$ of $T$-categories. What is not immediately clear is that the result may be equipped with the structure of a proof-relevant $\mathbb{D}$ algebra. However, some pullback lemmas show that $P(G * F) = G * (PF)$ so that in fact, by Lemma 2.4.5 (iii), we may endow $G * PF$ with the data of a split contraction such that the pullback square is a morphism of split contractions.

Without considering split contractions, that this pullback action of $T$-category morphisms is pseudo-functorial is a protracted calculation, but inevitably it reduces to mechanical universal property arguments. In order to claim that this renders $\mathrm{Alg}^{\mathrm{PR}}$ a pseudo-functor, however, we must demonstrate that the compositors and unitors

$$
\varphi_{G,G'}\colon \mathrm{Alg}^{\mathrm{PR}}G' \circ \mathrm{Alg}^{\mathrm{PR}}G \xrightarrow{\cong} \mathrm{Alg}^{\mathrm{PR}}(GG') \qquad \text{and} \qquad \varphi_{\mathbb{D}}\colon \mathrm{Alg}^{\mathrm{PR}}\,\mathrm{id}_{\mathbb{D}} \xrightarrow{\cong} \mathrm{id}_{\mathrm{Alg}^{\mathrm{PR}}\mathbb{D}}
$$

have components which descend to morphisms of split contractions.

Ultimately this reduces essentially to the following question. Suppose in the diagram of pullbacks below in an arbitrary category that $r$ has right lifting data with respect to some class of morphisms. Is the right lifting data endowed on $r'$ via $f, g$, and therefrom to $r''$ via $h, k$, the same as the right lifting data endowed on $r''$ via $fh, gk$?

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "r''"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[r, "f"] \arrow[d, "r'"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r"] \\ X \arrow[r, "k"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

As the reader may verify, because the transfer of right lifting data happens by composition and the universal property of pullbacks, this form of coherence holds and therefore the unitors and compositors of $\mathrm{Alg}^{\mathrm{PR}}$ have components which are morphisms of proof-relevant algebras. ■

Equipped with this language, we may seek to demonstrate that the inclusion of algebras into proof-relevant algebras is suitably natural.

**Lemma 5.3.4.** *The inclusion* $\mathrm{Alg}\,\mathbb{C} \rightarrowtail \mathrm{Alg}^{\mathrm{PR}}\mathbb{C}$ *is the component of a pseudo-natural transformation* $I\colon \mathrm{Alg} \Rightarrow \mathrm{Alg}^{\mathrm{PR}}$.

*Proof.* Recall Definition 2.2.10 so that we understand Alg to act on $T$-morphisms by pullback, entirely in the same manner as $\mathrm{Alg}^{\mathrm{PR}}$. Protracted if nevertheless mechanical universal property arguments show that there is at least a pseudo-natural transformation $\mathrm{Alg} \Rightarrow \mathrm{Alg}^{\mathrm{PR}}$ whose components are morphisms of $T$-$\mathrm{Cat}/\mathbb{C}$, and so the content of this claim reduces to the following general coherence question about the transfer of right lifting data.

Given an isomorphism $f$ and a conterminous morphism $g$, we wish to establish that endowing the isomorphism $g^*f$ with the data of right liftings generates the same right lifting data as first endowing $f$ with such data, and then using that to endow $g^*f$ with the data of right liftings. However, owing to the presence of universal properties in endowing $g^*f$ with data from that of $f$, these two processes may be seen to give the same result. Overall then, $I$ is pseudo-natural. ■

There is a final special case which is of interest in generating further examples of proof-relevant algebras.

**Lemma 5.3.5.** *Let* $\mathbb{C}$ *be a proof-relevant category. Then there is a fully-faithful, injective-on-objects functor* $\mathrm{Alg}\,\mathbb{C} \to \mathrm{Alg}^{\mathrm{PR}}1/\mathbb{C}$.

*Proof.* This lemma amounts to an observation stated in [Lei09, §10.2], but we frame it more generally here.

A $\mathbb{C}$-algebra $F\colon \mathbb{B} \to \mathbb{C}$ is a discrete opfibration, and so in particular its domain leg $d^{\mathbb{B}}\colon B^1 \to B^0$ is a pullback of the leg $d^{\mathbb{C}}\colon C^1 \to C^0$. But this latter leg is equipped with the data of a split contraction, so by Lemma 2.4.5 (iii) we may endow $d^{\mathbb{B}}$ with the data of a split contraction *such that* the resulting square is a morphism of split contractions. That is, $F$ is canonically an object of $\mathrm{Alg}^{\mathrm{PR}}1/\mathbb{C}$. The resulting assignment is straightforwardly injective-on-objects, and the extension to a functor may be readily verified to be fully-faithful. ■

## 5.4. Proof-relevant algebras shaped by $\omega$-categories

In [Lei04, §4], Leinster hints that the following, tantalising construction is possible.

> *"[...] there is a $T$-multicategory* $\mathbb{M}$AP *such that a* $\mathbb{M}$AP*-algebra is a pair* $(X,Y)$ *of strict $\omega$-categories together with a strict $\omega$-functor $X \to Y$"*

In this section we will generalise the spirit of this suggestion by defining a functor $\mathbb{S}\colon \omega\mathrm{Cat} \to T\text{-}\mathrm{Cat}$ such that the (proof-relevant) algebras for $\mathbb{S}\mathcal{C}$ are (proof-relevant) algebras which are "shaped like" the $\omega$-category $\mathcal{C}$.

This construction is central to the definitions we have presented thus far, and in Table 5.4.5 we demonstrate how our definitions of proof-relevant category and proof-relevant functor arise from this framework. In addition, we suggest several further definitions for the reader to explore.

The functor $\mathbb{S}$, which we will suggestively call shape, arises from rather more general considerations which we outline in Appendix B. We invite the reader to explore what we hope will prove to be part of a wider edifice emerging from the confluence of the theory of $T$-categories, enrichment, and algebras. In case that this excursus is not of interest, we will directly recall the specialisation of the Grothendieck construction result Theorem B.3.15 here.

In order to define this shape functor, it will be convenient to view $\omega$-categories as categories enriched in $\omega\mathrm{Cat}$—see Definition/Notation 2.5.3.

**Lemma 5.4.1** (Lemma B.3.17). *There is a functor $\mathbb{S}\colon \omega\mathrm{Cat} \to T\text{-}\mathrm{Cat}$ which sends an $\omega$-category $(\operatorname{ob}\mathcal{C}, \mathcal{C}(-,-), \mathrm{id}, \circ)$ to a $T$-category $\mathbb{S}\mathcal{C}$ on the graph*

$$\begin{tikzcd}
\coprod_{\operatorname{ob}\mathcal{C}} 1 & \coprod_{a,b\in\operatorname{ob}\mathcal{C}} \mathcal{C}(a,b)\times T1 \arrow[l, dashed, "c"'] \arrow[r, dashed, "d"] & T\left(\coprod_{\operatorname{ob}\mathcal{C}} 1\right) \\
1 \arrow[u, "\iota_b"] & \mathcal{C}(a,b)\times T1 \arrow[l, "!"] \arrow[r, "\pi"'] \arrow[u, "\iota_{a,b}"] & T1 \arrow[u, "T\iota_a"']
\end{tikzcd} .$$

*In addition, there is right adjoint functor $\mathrm{Alg}_{\omega\mathrm{Cat}}(\mathcal{C}) \to \mathrm{Alg}(\mathbb{S}\mathcal{C})$, and both functors of the adjunction are faithful.* ■

The "in addition" part of the above lemma justifies this entire endeavour, as we shall now explain. The notion of an algebra for an $\omega$-category is hopefully a familiar device, though perhaps dressed in unfamiliar terminology. While we not attempt to make the following rigorous, we will nevertheless make a moral claim: $\omega\mathrm{Cat}$ is cartesian closed, and in this way there is an equivalence

$$\mathrm{Alg}_{\omega\mathrm{Cat}}(\mathcal{C}) \simeq \omega\mathrm{Cat}\text{-}\mathrm{Cat}(\mathcal{C}, \omega\mathrm{Cat}) \simeq \omega\mathrm{Cat}_{\mathrm{large}}(\mathcal{C}, \omega\mathrm{Cat}) .$$

That is, we claim that morally an algebra for an $\omega$-category is the same thing as an $\omega$-(co)presheaf. It functorially supplies an $\omega$-category for each object of $\mathcal{C}$, an $\omega$-functor for each morphism of $\mathcal{C}$, an $\omega$-natural transformation for each 2-morphism of $\mathcal{C}$, and so on. Any equations in $\mathcal{C}$ become equations between the various transfors in $\omega$Cat, and in this way we model a copy of $\mathcal{C}$ but with potentially a great deal of internal data.

Thus, Lemma 5.4.1 above says, in particular, that among the examples of algebras for $\mathbb{S}\mathcal{C}$ must be faithfully those of $\mathcal{C}$ itself—things shaped entirely like $\mathcal{C}$ and obeying the same equations as those internal to $\mathcal{C}$. Moreover, this is comparison is an adjunction which we take as an indication of a great deal of structure in this transfer.

Let us record the overall combination of this discussion with Lemma 5.3.1.

**Lemma 5.4.2.** *For every $\omega$-category $\mathcal{C}$, we have the following comparisons of algebras.*

$$\begin{tikzcd} \mathrm{Alg}_{\omega\mathrm{Cat}}(\mathcal{C}) \arrow[r, bend right, "\mathit{faith.}"'] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathbb{S}\mathcal{C}) \arrow[l, bend right, "\mathit{faith}"'] \arrow[r, tail, "\mathit{f.f.}"] & \mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}\mathcal{C}) \end{tikzcd} \qquad \blacksquare$$

*Regret* 5.4.3. At this "point-wise" level we do not enjoy the full wealth that such a statement might offer. What we wish to know is that the above functors are all suitably natural. Should this be the case, then all equational aspects of $\omega$-categories would be transferable to algebras and proof-relevant algebras over $\mathbb{S}$. That is, given an $\omega$-functor $f\colon \mathcal{C} \to \mathcal{D}$, should the following square be commutative

$$\begin{tikzcd} \mathrm{Alg}_{\omega\mathrm{Cat}}(\mathcal{D}) \arrow[r, "\mathrm{faith.}"] \arrow[d, "\mathrm{Alg}_{\omega\mathrm{Cat}}(f)"'] & \mathrm{Alg}(\mathbb{S}\mathcal{D}) \arrow[d, "\mathrm{Alg}(\mathbb{S}f)"] \\ \mathrm{Alg}_{\omega\mathrm{Cat}}(\mathcal{C}) \arrow[r, "\mathrm{faith.}"'] & \mathrm{Alg}(\mathbb{S}\mathcal{C}) \end{tikzcd},$$

then we may reasonably claim that $\mathrm{Alg}(\mathbb{S}f)$ does to $\mathbb{S}\mathcal{D}$-algebras what $f$ does to $\mathcal{D}$-algebras.

For example, suppose that our $\omega$-functor $f$ is the inclusion of the free 2-category on a monad Mnd into the Schanuel-Street 2-category of the free adjunction Alg. Should the above diagram be commutative, then we could claim that $\mathrm{Alg}(\mathbb{S}f)$ extracts from any adjunction-algebra the underlying monad-algebra. ◁

The above remark therefore motivates us to make the following suggestion, for which regrettably we do not produce a proof.

**Conjecture 5.4.4.** *The functors of Lemma B.3.17 are natural with respect to morphisms of $\omega$-categories.*

To conclude this section, let us give some examples of $\omega$-categories $\mathcal{C}$, along with their algebras, the nature of proof-relevant algebras for $\mathbb{S}\mathcal{C}$, and some examples of these. Where we have written a question mark in the below table we mean to propose a definition, and consequently we omit examples. Finally, for brevity we use $F^\omega$ to denote the free strict $\omega$-category functor $\widehat{\mathbb{G}} \to \omega\mathrm{Cat}$.

Table 5.4.5: Examples of the shape functor, noting Lemma 5.4.2

| $\omega$-category $\mathcal{C}$ | $\mathcal{C}$-algebras are | Proof-relevant $\mathbb{S}\mathcal{C}$-algebras are | Examples of proof-relevant $\mathbb{S}\mathcal{C}$-algebras |
|---|---|---|---|
| $1 = \{\cdot\}$ | $\omega$-categories | proof-relevant categories (Section 3) | monoidal categories, bicategories, tricategories, algebras for contractible operads, … |
| $2 = \{\cdot \leftarrow \cdot\}$ | $\omega$-functors | proof-relevant functors (Section 4) | pseudo-functors, anafunctors |
| $F^{\omega}2_1 = \left\{ \begin{tikzcd} \cdot & \cdot \arrow[l, bend right] \arrow[l, bend left] \arrow[l, phantom, "\Uparrow"] \end{tikzcd} \right\}$ | $\omega$-natural transformations | proof-relevant 1-transfors? | |
| $3 = \left\{ \begin{tikzcd} \cdot & \cdot \arrow[l] \\ & \cdot \arrow[u] \arrow[ul] \arrow[ul, phantom, "=", near start, shift left=2] \end{tikzcd} \right\}$ | $\omega$-functor composition | proof-relevant functor composition (Section 5.5) | … |
| Adj (Schanuel-Street) | 1-adjunctions of $\omega$-functors | proof-relevant adjunctions? | |
| $\mathrm{Mnd} = \left\{ \begin{tikzcd} \cdot \arrow[loop above, "\Delta_+"] \end{tikzcd} \right\}$ | 1-monads of $\omega$-categories | proof-relevant monads? | |

## 5.5. Application: compositions of proof-relevant functors

As a final application of the machinery of Lemma 5.4.2, let us consider a possible prescription for composites of proof-relevant functors. That is, let us examine proof relevant algebras of the $\omega$-category containing a commutative triangle, $3 :\equiv 2 +_1 2$.

We choose to spare the reader from encountering the details of constructing $\mathbb{S}3$ as indicated in Lemma 5.4.1[8], and instead elaborate the result. A proof-relevant algebra for $\mathbb{S}3$ comprises the following structures.

(i) Three proof-relevant categories, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, and three proof-relevant functors $H\colon \mathbb{C} \to \mathbb{B}$, $G\colon \mathbb{B} \to \mathbb{A}$, and $F\colon \mathbb{C} \to \mathbb{A}$. Let us set some notation.

$$\begin{tikzcd}
A & P \arrow[l, "c^{\mathbb{A}}"'] \arrow[r, "d^{\mathbb{A}}"] & TA \\
 & G \arrow[ul, "\tau^G"] \arrow[dr, "\sigma^G"] & \\
B & Q \arrow[l, "c^{\mathbb{B}}"'] \arrow[r, "d^{\mathbb{B}}"'] & TB \\
 & H \arrow[ul, "\tau^H"] \arrow[dr, "\sigma^H"] & \\
C & R \arrow[l, "c^{\mathbb{C}}"] \arrow[r, "d^{\mathbb{C}}"'] & TC
\end{tikzcd}
\qquad
\begin{tikzcd}
A & P \arrow[l, "c^{\mathbb{A}}"'] \arrow[r, "d^{\mathbb{A}}"] & TA \\
 & F \arrow[ul, "\tau^F"] \arrow[dr, "\sigma^F"] & \\
C & R \arrow[l, "c^{\mathbb{C}}"'] \arrow[r, "d^{\mathbb{C}}"'] & TC
\end{tikzcd}$$

(ii) A 2-morphism $\mathsf{bincomp}\colon G \circ H \to F$ in $\mathrm{Span}(\widehat{\mathbb{G}}, T)$, where the former globular set is the vertex of the following span over $A$ and $TC$.

$$\begin{tikzcd}
 & G \circ H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TH \arrow[r, "T\sigma^H"] \arrow[d, "T\tau^H"] & T^2C \arrow[r, "\mu_C"] & TC \\
A & G \arrow[l, "\tau^G"] \arrow[r, "\sigma^G"'] & TB^0 & &
\end{tikzcd}$$

This map $\mathsf{bincomp}\colon G \circ H \to F$ is required to render commutative the following diagram.

$$\begin{tikzcd}
G \circ \mathbb{B} \circ H \arrow[r, shift left, "G \circ \square^{\tau}"] \arrow[r, shift right, "\square^{\sigma} \circ H"'] & G \circ H \arrow[r, "\mathsf{bincomp}"] & F
\end{tikzcd}$$

*Remark* 5.5.1. The reader who prefers the language of bimodules may note that we are asking something on the way to requiring that the underlying bimodule of $F$ is a tensor $G \otimes_{\mathbb{B}} H$ of the underlying bimodules of $G$ and $H$. While we do not require that $F$ be universal in any way, this is nevertheless an especially evocative presentation for it renders reasonable our earlier remarks that each proof-relevant category itself should serve as an identity proof-relevant functor—if we are to believe notation and analogies, then $G = G \otimes_{\mathbb{B}} \mathbb{B}$. ◁

The previous remark suggests that we investigate our intuitions and prove that *a* composite of a proof-relevant functor $G\colon \mathbb{B} \to \mathbb{C}$ with the canonical proof-relevant functor $\mathbb{B}$ on $\mathbb{B}$ of Section 4.2.1 is given by $G$ itself. That is, we wish to show that the

[8]and spare the author from recounting these

triple $(G, G, \mathbb{B})$ forms a proof-relevant algebra for $\mathbb{S}3$. We might formulate this as the following lemma, for instance.

**Lemma 5.5.2.** *Fix a proof-relevant functor $G \colon \mathbb{B} \to \mathbb{C}$. Then there is a proof-relevant algebra for $\mathbb{S}3$ whose composable proof-relevant functors are $G$ and $\mathbb{B}$, and whose composed proof-relevant functor is $G$.*

*Proof.* While it is possible to give a direct proof for this lemma, in fact this lemma is already proven for us.

Observe that there is a certain functor $s^0 \colon 3 \to 2$ which sends the first of the composable arrows to the identity, and the next two to the free arrow of 2. The induced functor $\mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}s^0) \colon \mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}2) \to \mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}3)$ which acts by pullback precisely sends a proof-relevant functor $G \colon \mathbb{B} \to \mathbb{C}$ to the triple $(G, G, \mathbb{B})$—that is, $G$ is a composite of $G$ and $\mathbb{B}$. ■

As we see then we could prove several lemmas of this form for composites of proof-relevant functors, but in some ways this misses a crucial aspect of composition: the results are somehow essentially unique. There is no literal sense, in the category $\mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}3)$, that all objects are isomorphic—the morphisms are far too strict for that. Nevertheless, there is an appropriate sense in which all proof-relevant $\mathbb{S}3$ algebras whose underlying projections to $\mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}2) \times_{\mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}1)} \mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}2)$ match, are equivalent. Said another way, we claim that there is a sense in which any two composites $F, F'$ of the same two proof-relevant functors $G$ and $H$ are equivalent.

To see why this might be, let us fix a 0-cell $c$ of $\mathbb{C}$. From any values of this 0-cell under $F$ and $F'$ as below-left, and any values under $H$ and $G$ successively as below-right

$$\begin{matrix} a \overset{f}{\mapsfrom} c \\ a' \overset{f'}{\mapsfrom} c \end{matrix} \qquad \text{and} \qquad \tilde{a} \overset{g}{\mapsfrom} b \overset{h}{\mapsfrom} c\ ,$$

we obtain parallel proofs $\mathsf{bincomp}(g, h)$ and $f$ in $F$, and parallel proofs $\mathsf{bincomp}'(g, h)$ and $f'$ in $F'$. In the usual manner then (e.g. Lemma 3.2.7) this induces equivalences between the values $a \simeq \tilde{a}$ and $\tilde{a} \simeq a'$. Because equivalences compose (Lemma 3.2.13 (i)), we therefore generate an overall equivalence $a \simeq a'$. There is always at least one way to obtain values for a 0-cell under $H$ and $G$, so there is always at least one equivalence between any two values of $F$ and $F'$ on the same 0-cell.

Although the arguments may become more involved as we increase the dimensions of the cells, it is nevertheless hoped that by conjugating by appropriate equivalences which arise in this manner we will be able to deduce that all values of $F$ and $F'$ for the same pasting are equivalent, and therefore that composition of proof-relevant functors is appropriately weakly determined.

Let us end therefore with the following two suggestions of the validity of this notion of composition. We suggest that composites in this sense suitably generalise those of an important base of comparison, viz., the generalised anafunctors.

**Conjecture 5.5.3.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be $\omega$-categories viewed as proof-relevant categories, and let $H: \mathcal{C} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ be generalised anafunctors (Definition 4.2.4). Recall that each generalised anafunctor $G, H$ gives a proof-relevant functor $G^a, H^a$ by Lemma 4.2.7. In this way, a composite for $G^a$ and $H^a$ is given by $(G \cdot H)^a$—the proof-relevant functor associated to the composite of generalised anafunctors (Remark 4.2.6).*

Finally we suggest that this notion of composition bears the hallmarks of weak composition in some putative, future proof-relevant category of proof-relevant categories. In particular we suggest that two composites of the same two proof-relevant functors give rise to higher cells between them—higher cells we hope may one day be seen to be the mediating cells of an equivalence in analogy with Lemma 3.2.7.

**Conjecture 5.5.4.** *Given two composites $F$ and $F'$ of the same two proof-relevant functors $H$ and $G$, there are proof-relevant 1-transfors, that is algebras for $\mathrm{Alg}^{\mathrm{PR}}(\mathbb{S}F^{\omega}2_1)$, symmetrically $F \Rightarrow F'$ and $F' \Rightarrow F$.*

# A. The finitary monadicity of proof-relevant fibrations of $T$-graphs

In this appendix we will give a proof of the following theorem.

**Theorem 5.2.3.** *The functor* $V : (T\text{-Grph}/C)^{\mathrm{PRF}} \to T\text{-Grph}/C$ *is monadic and finitary for every* $T$*-graph* $C$*.*

While it certainly may be possible to give the left adjoint "all at once", we take inspiration from the work of [Cheo3a] and decompose the putative adjunction between $(T\text{-Grph}/C)^{\mathrm{PRF}}$ and $T\text{-Grph}/C$ into a string of monadic, finitary adjunctions

$$\begin{tikzcd} \cdots & (T\text{-Grph}/C)_1^{\mathrm{PRF}} \arrow[r, bend right, "V_1"'] \arrow[r, phantom, "\perp"] & (T\text{-Grph}/C)_0^{\mathrm{PRF}} \arrow[l, bend right, "F_1"'] \arrow[r, bend right, "V_0"'] \arrow[r, phantom, "\perp"] & T\text{-Grph}/C \arrow[l, bend right, "F_0"'] \end{tikzcd} ,$$

From these adjunctions we will be able to construct the desired adjunction

$$\begin{tikzcd} (T\text{-Grph}/C)^{\mathrm{PRF}} \arrow[r, bend right, "V"'] \arrow[r, phantom, "\perp"] & T\text{-Grph}/C \arrow[l, bend right, "F"'] \end{tikzcd}$$

essentially by passing to the limit of the above string and using properties of locally finitely presentable categories.

Let us sketch now our overall plan for the proof of Theorem 5.2.3.

*Proof* (Sketch). To prove Theorem 5.2.3 we will:

1. realise $(T\text{-Grph}/C)^{\mathrm{PRF}}$ as the limit over the $\omega^{\mathrm{op}}$ diagram of the functors $V_n$,
2. directly prove that each $V_n$, as well as $V$, satisfies the all-but-adjoint criteria of monadicity: they are conservative and their domains have, and they preserve, coequalisers of split pairs,
3. directly prove that each $V_n$, as well as $V$, creates filtered colimits so that each functor above is finitary,
4. directly construct left adjoints $F_n \dashv V_n$ so that each $V_n$ is finitary and monadic,

With all of that achieved, $(T\text{-Grph}/C)^{\mathrm{PRF}}$ will therefore be an $\omega^{\mathrm{op}}$ shaped limit of isofibrations and so a pseudo-limit. As such, $V$—as one of the projection legs—will be continuous. The finitary monadicity of each $V_n$ will be enough to conclude that the entire diagram occurs over locally finitely presentably categories and finitary functors, which properties will transfer to the limit. Thus $V$, a finitary and continuous functor between locally finitely presentable categories, will satisfy the adjoint functor theorem, whence the left adjoint $F \dashv V$. We will have already determined $V$ to satisfy the rest of the monadicity criteria in (2) above, so that overall $V$ is finitary and monadic. ■

The various aspects of our proof strategy of Theorem 5.2.3 are addressed individually in the below sections. We begin in Appendix A.1 by laying out the definitions of the intermediate categories $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$ we displayed above and conclude (1). Then, in the same section, we prove that $V$ and all $V_n$ are finitary and create co-equalisers of split pairs which addresses (2) and (3). Then in Appendix A.2 we tackle the most involved aspect of this proof, viz., the construction of a left adjoints $F_n$ to each functor $V_n$ so that we have (4) above. Finally in Appendix A.3, with all terms and supporting ideas having been made rigorous, we give a complete proof for Theorem 5.2.3.

For context in the coming sections let us fix $T$ to be the free strict $\omega$-category monad on $\widehat{\mathbb{G}}$. Although certainly many of our results are divorced from this specificity, our only application is to such and so we wish to ease the burden on the burden on the reader by staying concrete.

*Remark.* Our work here is noticeably more involved than the situation addressed by the lucid and comprehensive account of [Che03a]. There the author constructs a similar string of left adjoints, but the categories of interest are simpler in a few ways.

First, while we are interested in proof-relevant fibrations, [Che03a]—as a consequence of meeting the theory of [Lei09, §9.1]—is interested in "contraction fibrations". The difference is that where we require lifts against $\partial_n \rightarrowtail 2_n$ for all $n$, in the framework of Leinster lifts are only required for positive $n$. Noting Remark 2.4.2, this requires us to solve a different type of problem for the dimension zero case, viz., freely adjoining a splitting. Next, while we are interested in $T$-graphs fibred over a general $T$-graph $C$, [Che03a] has interest only in $T$-graphs $A$ fibred over the terminal $T$-graph $1 \leftarrow T1 = T1$. Finally, we are interested in general $T$-graphs $A$ whereas the work there constrains all $T$-graphs to have $A^0 = 1$.

The first difference is somehow inessential, we uniformly adjoin solutions to lifting problems regardless of dimension so this qualitative difference is imperceptible. The second difference introduces some extra bookkeeping that would otherwise be invisible, but proves to be of no major obstruction. While last difference might seem to be the most trivial, in fact there is a large qualitative difference in the nature of the left adjoints. In the case where all $T$-graphs have $A^0 = 1$, as hinted at in [Lei09, §9.1] and accounted for in [Che03a], solutions to lifts against $\partial_n \rightarrowtail 2_n$ depend only on strictly lower dimensional cells and so there is a clear strategy to freely add such lifts. However, in the case of general $A^0$, the lifting problems introduce an element of feedback: in order to freely adjoin cells to $A^1$ to solve the lifting problem given in terms of $A^0$, we must additionally freely adjoin cells to $A^0$; but doing this requires us to adjoin more cells to $A^1$, and so on *ad infinitum*. That is, although lifting the restriction $A^0 = 1$ may be as a yoke shed, we constrain ourselves nevertheless by the introduction of a "least fixed-point" problem. ◁

## A.1. Decomposing the adjunction

Let us now give some meaning to the categories $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$ displayed above. Recall that $(T\text{-Grph}/C)^{\mathrm{PRF}}$ was defined to be the vertex of the below pullback in Cat.

$$\begin{tikzcd} (T\text{-Grph}/C)^{\mathrm{PRF}} \arrow[r, "V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-Grph}/C \arrow[d, "P"] \\ \mathrm{SCon}\,\widehat{\mathbb{G}} \arrow[r] & \mathrm{Ar}\,\widehat{\mathbb{G}} \end{tikzcd} \tag{A.1.1}$$

Let us therefore analogously define the categories $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$ to be "proof-relevant fibrations good up to dimension $n$".

**Definition A.1.2.** The category $\mathrm{SCon}_{\leq n}\,\widehat{\mathbb{G}}$ comprises those objects which are globular morphisms with chosen lifts against $\partial_k \rightarrowtail 2_k$ only for $k \leq n$, and whose morphisms are those of the arrow category which preserve these lifts (Definition 2.4.1). There are evident forgetful functors $\mathrm{SCon}_{\leq 0}\,\widehat{\mathbb{G}} \to \mathrm{Ar}\,\widehat{\mathbb{G}}$ and $\mathrm{SCon}_{\leq n+1}\,\widehat{\mathbb{G}} \to \mathrm{SCon}_{\leq n}\,\widehat{\mathbb{G}}$.

The category $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$ is inductively defined to be the vertex of the below-left (resp. below-right) pullback in Cat for $n = 0$ (resp. $n = k + 1$).

$$\begin{tikzcd} (T\text{-Grph}/C)^{\mathrm{PRF}}_0 \arrow[r, "V_0"] \arrow[d, "P_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & T\text{-Grph}/C \arrow[d, "P"] \\ \mathrm{SCon}_{\leq 0}\,\widehat{\mathbb{G}} \arrow[r] & \mathrm{Ar}\,\widehat{\mathbb{G}} \end{tikzcd} \qquad \begin{tikzcd} (T\text{-Grph}/C)^{\mathrm{PRF}}_{n+1} \arrow[r, "V_n"] \arrow[d, "P_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (T\text{-Grph}/C)^{\mathrm{PRF}}_n \arrow[d, "P_n"] \\ \mathrm{SCon}_{\leq n+1}\,\widehat{\mathbb{G}} \arrow[r] & \mathrm{SCon}_{\leq n}\,\widehat{\mathbb{G}} \end{tikzcd}$$

We extend our notation to the extra value of $n = -1$ by setting $\mathrm{SCon}_{\leq -1}\,\widehat{\mathbb{G}} = \mathrm{Ar}\,\widehat{\mathbb{G}}$ and $(T\text{-Grph}/C)^{\mathrm{PRF}}_{-1} = T\text{-Grph}/C$. ◃

We wish to highlight in particular that we thus defined a string of would-be right adjoints,

$$\cdots (T\text{-Grph}/C)^{\mathrm{PRF}}_1 \xrightarrow{V_1} (T\text{-Grph}/C)^{\mathrm{PRF}}_0 \xrightarrow{V_0} (T\text{-Grph}/C)^{\mathrm{PRF}}_{-1} = T\text{-Grph}/C\ .$$

We shall now show that each of these functors, and indeed also the functor of central concern $V\colon (T\text{-Grph}/C)^{\mathrm{PRF}} \to T\text{-Grph}/C$, enjoy the stronger-than-finitary property of creating filtered colimits and coequalisers of split pairs. Thus, after this work to conclude the theorem we must only provide a left adjoint $F \dashv V$.

**Lemma A.1.3** (All-but-adjoint monadicity of $V$). *Notwithstanding the data of a left adjoint, the forgetful functor $V\colon (T\text{-Grph}/C)^{\mathrm{PRF}} \to T\text{-Grph}/C$ meets the requirements of finitary monadicity.*

*(i) $V$ creates filtered colimits, and so is finitary; thus the category $(T\text{-Grph}/C)^{\mathrm{PRF}}$ has all filtered colimits.*

*(ii) $V$ is conservative and $(T\text{-Grph}/C)^{\mathrm{PRF}}$ has, and $V$ preserves, coequalisers of $V$-split pairs.*

*In addition the same is true of each functor* $V_n\colon (T\text{-Grph}/C)^{\mathrm{PRF}}_n \to (T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ *and each category* $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$.

In order to prove Lemma A.1.3 we will need to establish some technical lemmas about the nature of filtered colimits and finitely presented objects. First let us recall the key properties of filtered colimits in Set.

**Lemma A.1.4.** *Let* $F\colon I \to \mathrm{Set}$ *be a filtered diagram. Then,*

*(i) a colimit of* $F$ *is given by the set* $(\coprod_I Fi)/\approx$ *where the equivalence relation* $\approx$ *is defined by* $(i,x) \approx (i',x')$ *when and only when there exists some span* $i \to j \leftarrow i'$ *such that the images of* $x, x'$ *in* $Fj$ *are equal,*

*(ii) for finitely many elements* $\{(i_k,x_k)\}_{k\le n}$ *of this colimit, there exists a single* $i \in \mathrm{ob}\, I$ *and elements* $\{(i,y_k)\}_{k\le n}$ *such that* $[(i_k,x_k)] = [(i,y_k)]$ *for all* $k \le n$*, and*

*(iii) for finitely many elements* $\{(i,x_k)\}_{k\le n}$ *of this colimit we have* $[(i,x_1)] = \ldots = [(i,x_n)]$ *if and only if there is some* $i \to j$ *in* $I$ *such that all the* $x_k$ *have equal images in* $Fj$.

*Proof.* For (i) see for instance the notationally-matched result [Bor94a, Prop. 2.13.3].

Now suppose given finitely many objects $\{(i_k,x_k)\}_{k\le n}$ as in (ii) above. As $I$ is filtered, there is a cocone $i$ on the discrete finite diagram $\{i_k\}_{k\le n}$, and thus using the cospan $i_k \to i \leftarrow i$ we see that $[(i_k,x_k)] = [(i,y_k)]$ where $y_k$ is the image of $x_k$ under $i_k \to i$.

One direction of (iii) is immediate: suppose that all the $x_k$ have equal images in $Fj$ under $i \to j$, then the cospan $i \to j \leftarrow i$ witnesses all the equalities $[(i,x_k)] = [(i,x_{k'})]$ by definition. Conversely suppose that we have $[(i,x_k)] = [(i,x_{k'})]$ for all $k,k' \le n$, so that there are potentially $(n^2-n)$-many maps, say for each distinct pair $k \ne k'$ a pair of maps $m^k_{\{k,k'\}}, m^{k'}_{\{k,k'\}}\colon i \to j_{\{k,k'\}}$, such that in $Fj_{\{k,k'\}}$ we have the equality $F(m^k_{\{k,k'\}})x_k = F(m^{k'}_{\{k,k'\}})x_{k'}$ for each pair $x_k, x_{k'}$ . However, $I$ is filtered and so there is a cocone $j$ on the finite diagram $\{i \rightrightarrows j_{\{k,k'\}}\}$. We are thus assured by the commutativity of the cocone that all the $x_k$ have equal images in $Fj$ under the cocone morphism $F(i \to j)$. ∎

Next we wish to establish some facts about factorisations of maps into filtered colimits in arrow categories.

**Lemma A.1.5.** *Let* $\mathscr{C}$ *be a category. Suppose that* $F\colon I \to \mathrm{Ar}\,\mathscr{C}$ *is a filtered diagram, and that* $q\colon X \to Y$ *is an object of* $\mathrm{Ar}\,\mathscr{C}$ *such that both* $X$ *and* $Y$ *are finitely presented in* $\mathscr{C}$*. Let* $(\mathrm{colim}\, F, \iota)$ *be a colimit of* $F$ *created by the projections* $(-)^1, (-)^0\colon \mathrm{Ar}\,\mathscr{C} \to \mathscr{C}$*. For* $(f,g)\colon q \to \mathrm{colim}\, F$ *a morphism of* $\mathrm{Ar}\,\mathscr{C}$ *we have the following existence, improvement and refinement results for its factorisations through the diagram* $F$.

*(i) There exists a pair of factorisations in* $\mathscr{C}$,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_i"'] & & \mathrm{colim}(F^1) \\ & (Fi)^1 \arrow[ur, "\iota^1_i"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "g_j"'] & & \mathrm{colim}(F^0) \\ & (Fj)^0 \arrow[ur, "\iota^1_j"'] & \end{tikzcd},$$

*for some indices $i, j \in \operatorname{ob} I$.*

*(ii) For any pair of factorisations as in (i), there exists a cospan $i \longrightarrow \ell \longleftarrow j$ in $I$ such that we may obtain factorisations in $\mathscr{C}$ of $f$ and $g$ as*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f_i"'] \arrow[dr, dashed, "f_\ell"] & \operatorname{colim}(F^1) \\
(Fi)^1 \arrow[r, "F(i \to \ell)^1"'] & (F\ell)^1 \arrow[u, "\iota^1_\ell"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "g_j"'] \arrow[dr, dashed, "g_\ell"] & \operatorname{colim}(F^0) \\
(Fj)^0 \arrow[r, "F(j \to \ell)^0"'] & (F\ell)^0 \arrow[u, "\iota^1_j"']
\end{tikzcd}
,
$$

*through the same index $\ell \in \operatorname{ob} I$.*

*(iii) For any pair of factorisations as in (ii), there exists a morphism $c \colon \ell \longrightarrow k$ in $I$ such that we may obtain the following factorisation of $(f, g)$ in $\operatorname{Ar}\mathscr{C}$ through $(f_k, g_k)$.*

$$
\begin{tikzcd}
 & (F\ell)^1 \arrow[rr, "(Fc)^1"] & & (Fk)^1 \arrow[dl, "\iota^1_k"] \arrow[dd, "\vec{F}k"] \\
X \arrow[ur, "f_\ell"] \arrow[urrr, dashed, "f_k"] \arrow[rr, "f"] \arrow[dd, "q"'] & & \operatorname{colim}(F^1) \arrow[dd] & \\
 & (F\ell)^0 \arrow[rr] & & (Fk)^0 \arrow[dl, "\iota^0_k"] \\
Y \arrow[ur, "g_\ell"] \arrow[urrr, dashed, "g_k"] \arrow[rr, "g"'] & & \operatorname{colim}(F^0) &
\end{tikzcd}
$$

*(iv) There exists a factorisation $(f, g) = \iota_k(f_k, g_k)$ in $\operatorname{Ar}\widehat{\mathbb{G}}$, for some index $k \in \operatorname{ob} I$.*

*(v) For any pair factorisations as in (iv), $(f, g) = \iota_i(f_i, g_i)$ and $(f, g) = \iota_j(f_j, g_j)$, there is a third index $k \in \operatorname{ob} I$ and a span $i \xrightarrow{a} k \xleftarrow{b} j$ such that*

$$(Fa)(f_i, g_i) = (Fb)(f_j, g_j) \ .$$

*Proof.* One of our hypotheses is that the colimit of $F$ in the arrow category is computed coordinate-wise in the category $\mathscr{C}$. As such, because $X$ and $Y$ are assumed to be finitely presented, we may deduce (i) from the explicit formula for the filtered colimit of sets given in Lemma A.1.4 (i). Furthermore, given such factorisations, as $I$ is filtered and $\iota$ is a cocone we may deduce (ii) readily.

Suppose as in (iii) that we have factorisations $f_\ell$ and $g_\ell$ as present in the first commutative triangles in the top and bottom faces below-right. Note that we have not included the vertical morphism $\vec{F}\ell$ as we have no commutativity assumptions on it.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q"'] & \operatorname{colim}(F^1) \arrow[d, "\operatorname{colim}\vec{F}"] \\
Y \arrow[r, "g"'] & \operatorname{colim}(F^0)
\end{tikzcd}
\qquad
\begin{tikzcd}
& (F\ell)^1 \arrow[rr, "(Fc)^1"] \arrow[dr, "\iota^1_\ell"] & & (Fk)^1 \arrow[dl, "\iota^1_k"'] \arrow[dd, "\vec{F}k"] \\
X \arrow[ur, "f_\ell"] \arrow[rr, "f"] \arrow[dd, "q"'] & & \operatorname{colim}(F^1) \arrow[dd] & \\
& (F\ell)^0 \arrow[rr] \arrow[dr, "\iota^0_\ell"] & & (Fk)^0 \arrow[dl, "\iota^0_k"] \\
Y \arrow[ur, "g_\ell"] \arrow[rr, "g"'] & & \operatorname{colim}(F^0) &
\end{tikzcd}
$$

Observe that the two elements $(\vec{F}\ell)f_\ell$ and $g_\ell q$ of $\mathscr{C}(X,(F\ell)^0)$ have equal images in $\mathscr{C}(X,\operatorname{colim}F^0)$ under $(\iota^0_k)_*$ from the commutativity assumption above-left. But $X$ is finitely presented so that $\mathscr{C}(X,\operatorname{colim}F^0) \cong \operatorname{colim}\mathscr{C}(X,F^0)$ is a filtered colimit. Thus, by Lemma A.1.4 (iii) there must exist a morphism $c\colon \ell \to k$ such that $(\vec{F}\ell)f_\ell$ and $g_\ell q$ have equal images under $F^0(c)$. Certainly were $((Fc)^1 f_\ell, (Fc^0)g_\ell)$ to be a morphism in $\operatorname{Ar}\mathscr{C}$ it would satisfy $(f,g) = \iota_k((Fc)^1 f_\ell, (Fc^0)g_\ell)$. Thus to conclude (iii) it remains to show that $((Fc)^1 f_\ell, (Fc^0)g_\ell)$ is a morphism of the arrow category, but this is witnessed by the following reasoning.

$$(\vec{F}k)(Fc)^1 f_\ell = (Fc)^0(\vec{F}\ell)f_\ell = (Fc)^0 g_\ell q$$

The next claim (iv) is easily derived by applying (iii) to (ii) to (i). Let us thus address (v). To make efficient the proof, let us suppose that we have a pair of factorisations $\iota^1_i f_i = \iota^1_{i'} f_{i'}$ and prove that there is some $i \to \hat{\imath} \leftarrow i'$ such that the morphisms $f_i$ and $f_{i'}$ under the action of $F$ on this cospan. Should we achieve this, then given two factorisations, $(f,g) = \iota_i(f_i,g_i)$ and $(f,g) = \iota_j(f_j,g_j)$, we may apply this result to obtain cospans $i \to i' \leftarrow j$ and $i \to j' \leftarrow j$ under whose action respectively $f_i$ and $f_j$ agree, and $g_i$ and $g_j$ agree. Let us call $f_{i'}$ and $g_{j'}$ these composites, and note that we may apply (iii) to (ii) to this pair of factorisations to obtain (v).

Thus to conclude the proof it remains to give the cospan $i \to \hat{\imath} \leftarrow i'$. First we apply Lemma A.1.4 (ii) to the equal elements of $\mathscr{C}(X,\operatorname{colim}F^1) \cong \operatorname{colim}\mathscr{C}(X,F^1)$ to obtain morphisms $i \to \bar{\imath} \leftarrow i'$ and elements $\overline{f}_i, \overline{f}_{i'} \in \mathscr{C}(X,(F\bar{\imath})^1)$ agreeing with $f_i$ and $f_{i'}$ in the colimit. By Lemma A.1.4 (iii) there is therefore some $\bar{\imath} \to \hat{\imath}$ such that under the composite $i \to \hat{\imath} \leftarrow i'$ the morphisms $f_i$ and $f_{i'}$ are equal in $\mathscr{C}(X,(F\hat{\imath})^1)$ as desired. ■

At the risk of belabouring the utility of the properties of filtered colimits, we ask the reader to extend their patience for a final "filtered colimits build solutions to lifting problems" lemma. This time we return to the specific case of considering the category $\widehat{\mathbb{G}}$ and the free strict $\omega$-category monad $T$ thereupon.

*Remark* A.1.6. The statement of the following lemma takes for granted that filtered colimits of a functor $F\colon I \to T\text{-}\mathrm{Grph}/C$ are computed "coordinate-wise" in $T\text{-}\mathrm{Grph}/C$ as we now illustrate.

$$\begin{tikzcd}
\operatorname{colim}(V_nF)^0 \arrow[d, "\operatorname{colim} f^0"'] & \operatorname{colim}(V_nF)^1 \arrow[l, "\operatorname{colim}(V_nF)^\ell"'] \arrow[r, "\operatorname{colim}(V_nF)^r"] \arrow[d, "\operatorname{colim} f^1"'] & T\operatorname{colim}(V_nF)^0 \arrow[d, "T\operatorname{colim} f^0"] \\
C^0 & C^1 \arrow[l, "C^\ell"] \arrow[r, "C^r"'] & TC^0
\end{tikzcd}$$

By the proof of Lemma 5.2.6 we know that $T$-Grph$/C$ is the slice of the Artin gluing of a certain finitary functor, so that in fact filtered colimits are computed as above. Moreover, we may commute pullback and $T$ with the filtered colimit because the former is a left adjoint and the latter preserves such (Theorem 2.5.1). ◃

**Lemma A.1.7.** *Fix $n \in \mathbb{N}$, a filtered diagram $F\colon I \to T$-Grph$/C$, and a colimit $(\operatorname{colim} F, \iota)$ for that diagram. Suppose that for each $n$-dimensional lifting problem as below-left there exists an index $i \in \operatorname{ob} I$ such that the factored problem $(\partial\varphi, \Phi) = \iota_i(\partial\varphi_i, \Phi_i)$ below-right admits a solution $\kappa_i(\partial\varphi_i, \Phi_i)\colon 2_n \to (Fi)^1$.*

$$\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & \operatorname{colim} F^1 \arrow[d, "\operatorname{colim} PF"] \\
2_n \arrow[r, "\Phi"'] & \operatorname{colim}\left(C^1 \underset{TC^0}{\times} TF^0\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (Fi)^1 \arrow[dr, "\iota_i^1"] \arrow[d] & \\
\partial_n \arrow[ur, "\partial\varphi_i"] \arrow[rr, "\partial\varphi"] \arrow[dd, tail] & C^1 \underset{TC^0}{\times} T(Fi)^0 \arrow[dr] & \operatorname{colim} F^1 \arrow[d] \\
 & & \operatorname{colim}\left(C^1 \underset{TC^0}{\times} TF^0\right) \\
2_n \arrow[uur, "\Phi_i"] \arrow[urr, "\Phi"'] & &
\end{tikzcd}
\tag{A.1.8}$$

*Suppose further that whenever this is true, for all morphisms $a\colon i \to k$ of $I$ there is a solution $\kappa_k\big((PFa)(\partial\varphi_i, \Phi_i)\big)\colon 2_n \to (Fk)^1$ as below for the problem $(PFa)(\partial\varphi_i, \Phi_i)$. Suppose finally that $PFa$ is always a morphism of theses solutions, that is, we have the equality $(PFa)^1\kappa_i(\partial\varphi_i, \Phi_i) = \kappa_k\big((PFa)(\partial\varphi_i, \Phi_i)\big)$.*

$$\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi_i"] \arrow[d, tail] & (Fi)^1 \arrow[r, "(Fa)^1"] & (Fk)^1 \arrow[d, "\vec{F}k"] \\
2_n \arrow[r, "\Phi_i"'] \arrow[urr, dashed, "\kappa_k((PFa)(\partial\varphi_i, \Phi_i))" description] & C^1 \underset{TC^0}{\times} T(Fi)^0 \arrow[r] & C^1 \underset{TC^0}{\times} T(Fk)^0
\end{tikzcd}$$

*Under these assumptions, the choice of solution $\kappa_{\mathrm{colim}}(\partial\varphi, \Phi) :\equiv \iota_i^1\kappa_i(\partial\varphi_i, \Phi_i)$ to the original problem is well-defined.*

*Proof.* Suppose that there was a competing factorisation $(\partial\varphi, \Phi) = (P\iota_j)(\partial\varphi_j, \Phi_j)$ in $\operatorname{Ar}\widehat{\mathbb{G}}$, and a solution $\kappa_j(\partial\varphi_j, \Phi_j)\colon 2 \to (Fj)^1$. Lemma A.1.5 (v) assures us that there is some cospan $i \xrightarrow{a} k \xleftarrow{b} j$ such that $(PFa)(\partial\varphi_i, \Phi_i) = (PFb)(\partial\varphi_j, \Phi_j)$. From this we may argue as follows

$$\begin{aligned}
\iota_i^1 \kappa_i(\partial\varphi_i, \Phi_i) &= \iota_k^1 (Fa)^1 \kappa_i(\partial\varphi_i, \Phi_i) && \text{(cocone)} \\
&= \iota_k^1 \kappa_k\big((PFa)(\partial\varphi_i, \Phi_i)\big) && \text{(assumption)} \\
&= \iota_k^1 \kappa_k\big((PFb)(\partial\varphi_j, \Phi_j)\big) && \text{(Lemma A.1.5 (v))} \\
&= \iota_k^1 (Fb)^1 \kappa_j(\partial\varphi_j, \Phi_j) && \text{(assumption)} \\
&= \iota_j^1 \kappa_j(\partial\varphi_j, \Phi_j) && \text{(cocone)}
\end{aligned}$$

so that our choice of solution is well-defined. ■

We are finally ready to give a proof of Lemma A.1.3. For clarity we have addressed the two claims in separate proofs.

*Proof* (Lemma A.1.3 (i)). Suppose that $F\colon I \to (T\text{-Grph}/C)^{\mathrm{PRF}}$ is filtered, and fix a colimit $(\operatorname{colim} VF, \iota)$ in $T$-Grph$/C$. Note that the hypotheses of Lemma A.1.7 are satisfied. For every lifting problem against $\operatorname{colim} PVF$, every index $i \in \operatorname{ob} I$ through which it factors—and there is at least one by Lemma A.1.5 (iv)—admits a solution $\kappa_i(\partial\varphi_i, \Phi_i)$ for each $Fi$ is an object of $(T\text{-Grph}/C)^{\mathrm{PRF}}$. In addition, every morphism $PFa$ is a morphism of split contractions and so preserves solutions. Thus, by Lemma A.1.7 there is *a* way to endow $\operatorname{colim} PVF$ with the data of solutions to lifting problems in all dimensions.

Let us now argue that this way is uniquely determined subject to the constraint that the solutions to lifting problems in dimension $n$ are compatible with the rest of the data of $F$ and the cocone $\iota$.

First we contend that for the split contraction data on $\operatorname{colim} PVF$, the components of the cocone each descend to a morphism of split contractions. That is, suppose that we had a lifting problem $(\partial\varphi_i, \Phi_i)$ as in (A.1.8). For $\iota_i$ to descend to a morphism of split contractions we must demonstrate the equality

$$\iota_i^1 \kappa_i(\partial\varphi_i, \Phi_i) = \kappa_{\mathrm{colim}}\big((P\iota_i)(\partial\varphi_i, \Phi_i)\big) ,$$

but of course this is precisely our definition of $\kappa_{\mathrm{colim}}$, where we have instead begun with $(\partial\varphi_i, \Phi_i)$ and observed that it is a factorisation of $(\partial\varphi, \Phi) :\equiv (P\iota_i)(\partial\varphi_i, \Phi_i)$ through $PFi$ as rendered *ibid.*

Outstanding in our claim that $V$ creates filtered colimits then is a proof of uniqueness. Certainly all the data but the lifts are fixed by the definition of $V$, and so it remains only to establish that the definition of $\kappa_{\mathrm{colim}}$ we have given is the unique such which is compatible with the rest of the data. Again, however, any choice of lifts must in particular allow for $\iota$ to descend to a morphism of split contractions, from which we see that it must agree with our definition of $\kappa_{\mathrm{colim}}$ precisely; all lifting problems factor through some $i \in I$ as we have established. Thus $V$ creates filtered colimits.

This argument may be adapted to give the proof that each $V_n$ creates filtered colimits. We must argue by induction: for $n = 0$, the above argument goes through essentially unchanged for filtered $F\colon I \to (T\text{-Grph}/C)_0^{\mathrm{PRF}}$ except constrained to zero-dimensional lifting problems. In the case of positive $n$, our inductive hypothesis informs us in part that $(T\text{-Grph}/C)_{n-1}^{\mathrm{PRF}}$ has filtered colimits, and so once more we may run the above argument for filtered $F\colon I \to (T\text{-Grph}/C)_n^{\mathrm{PRF}}$ except here constrained to $n$-dimensional lifting problems, and with the observation that the presence of solutions to lower-dimensional problems does not impact any of the claims. ■

*Proof* (Lemma A.1.3 (ii)). Let $u, v\colon A \rightrightarrows B$ be a parallel pair in $(T\text{-Grph}/C)^{\mathrm{PRF}}$, and suppose we have the data $(q\colon VB \to Q,\ s,\ t)$ of a $V$-split pair in $T\text{-Grph}/C$. In order to show that $(T\text{-Grph}/C)^{\mathrm{PRF}}$ has, and $V$ preserves, a coequaliser for this $V$-split pair we must induce a split contraction structure on $PQ$ for which $q\colon VB \to Q$ is a coequaliser in $(T\text{-Grph}/C)^{\mathrm{PRF}}$.

This $V$-split pair is preserved by all functors, and in particular by the functor $P\colon T\text{-Grph}/C \to \mathrm{Ar}\,\widehat{\mathbb{G}}$ of Definition 5.1.1. So it is tempting to argue that the data of the split contraction on $PB$ may be used to induce the data of a split contraction on $PQ$ via the retract. However, as is easily checked, this does not in general confer either the section or the retract with the property of preserving the chosen lifts—essentially because the associated idempotent is not necessarily the identity. We must therefore be more sophisticated in our approach.

Instead, following the theory of [BG16], let us argue by means of properties of cofibrantly generated algebraic weak factorisation systems (AWFS). By Lemma 2.4.4, the functor $U\colon \mathrm{SCon}\,\widehat{\mathbb{G}} \to \mathrm{Ar}\,\mathbb{G}$ is monadic and finitary, and $\mathrm{Ar}\,\widehat{\mathbb{G}}$ is locally finitely presentable. Observe that the pullback (A.1.1) in Cat defining $(T\text{-Grph}/C)^{\mathrm{PRF}}$ matches the hypothesis of Lemma 2.1.4 so that $V$ is conservative, and $(T\text{-Grph}/C)^{\mathrm{PRF}}$ has, and $V$ preserves coequalisers of $V$-split pairs.

Again this argument may be adapted for each $V_n$ by induction as follows. Let $\mathrm{ob}\,\mathcal{J}_n = \{\partial_n \rightarrowtail 2_n\}$ be the discrete category whose sole object is the boundary inclusions of dimension $n$. Observe moreover that by the definition of $\partial_{n+1} = 2_n +_{\partial_n} 2_n$ in fact there is an evident inclusion $\mathcal{J}_n \to \mathrm{SCon}_{\leq n-1}\,\widehat{\mathbb{G}}$ for each $n$. Should we establish that each $\mathrm{SCon}_{\leq n}\,\widehat{\mathbb{G}} \to \mathrm{SCon}_{\leq n-1}\,\widehat{\mathbb{G}}$ is finitary and monadic, with each $\mathrm{SCon}_{\leq n-1}\,\widehat{\mathbb{G}}$ locally finitely presentable, then by induction and Lemma 2.1.4 applied to the pullbacks in Definition A.1.2 the result follows.

When $n = 0$, $\mathrm{SCon}_{\leq -1}\,\widehat{\mathbb{G}} = \mathrm{Ar}\,\widehat{\mathbb{G}}$ is locally finitely presentable, and so [BG16, Prop. 16] gives a $\mathcal{J}_0$-cofibrantly generated AFWS whose finitary[9] monad $R_0$ yields the equivalence $\mathrm{SCon}_{\leq 0}\,\widehat{\mathbb{G}} \simeq (\mathrm{SCon}_{\leq -1}\,\widehat{\mathbb{G}})^{R_0}$. Thus $\mathrm{SCon}_{\leq 0}\,\widehat{\mathbb{G}} \to \mathrm{SCon}_{\leq -1}\,\widehat{\mathbb{G}}$ is monadic and finitary and $\mathrm{SCon}_{\leq 0}\,\widehat{\mathbb{G}}$ is locally finitely presentable [AR94, Rem. 2.75]. Inductively then, using each inclusion $\mathcal{J}_n \to \mathrm{SCon}_{\leq n-1}\,\widehat{\mathbb{G}}$ in turn, each $\mathrm{SCon}_{\leq n-1}\,\widehat{\mathbb{G}}$ is locally finitely presentable and each $\mathrm{SCon}_{\leq n}\,\widehat{\mathbb{G}} \to \mathrm{SCon}_{\leq n-1}\,\widehat{\mathbb{G}}$ is finitary and monadic. ■

[9]See the proof of Lemma 2.4.4 for a discussion of this claim, not explicit in the cited proposition.

## A.2. The string of left adjoints

In this section we wish to construct, for a fixed but arbitrary $n \in \mathbb{N}$, a left adjoint functor $F_n \dashv V_n$.

The construction of such an adjoint amounts to freely adding to diagrams, in $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ as below-left, the data of solutions of lifts against the boundary inclusions $\partial_n \rightarrowtail 2_n$ to the map indicated below-right. We have additionally annotated the objects and morphisms by a subscript $(k)$, but for we defer explaining this move for two paragraphs.

$$\begin{tikzcd}
A^0_{(k)} \arrow[d, "f^0_{(k)}"'] & A^1_{(k)} \arrow[l, "A^\ell_{(k)}"'] \arrow[r, "A^r_{(k)}"] \arrow[d, tail, "f^1_{(k)}"] & TA^0_{(k)} \arrow[d, "Tf^0_{(k)}"] \\
C^0 & C^1 \arrow[l, "C^\ell"] \arrow[r, "C^r"'] & TC^0
\end{tikzcd} \tag{A.2.1.a}$$

$$\begin{tikzcd}
A^1_{(k)} \arrow[dr, dashed, "Pf_{(k)}"] \arrow[drr, bend left, "A^r_{(k)}"] \arrow[ddr, bend right, "f^1_{(k)}"'] & & \\
& C^1 \underset{TC^0}{\times} TA^0_{(k)} \arrow[r, "\pi_A"] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & TA^0_{(k)} \arrow[d, "Tf^0_{(k)}"] \\
& C^1 \arrow[r, "C^r"'] & TC^0
\end{tikzcd} \tag{A.2.1.b}$$

When $n = 0$ we may be seen to be asking for the free split epimorphism on $PA$, and for positive $n$ we wish to freely add $n$ cells formally labelled by lifting problems. Both of these may readily be achieved by modifying $A^1_{(k)}$ in the appropriate dimension, but should we perform these constructions we will quickly run into a problem: to where in $A^0_{(k)}$ should we map these new cells of $A^1_{(k)}$? Free-ness demands that we introduce no new equations, and so we must adjoin new cells to $A^0_{(k)}$ to serve as the images of the new cells of $A^1_{(k)}$ under $A^\ell_{(k)}, A^r_{(k)}$. But lo, any change to $A^0_{(k)}$ changes $Pf_{(k)}$ and thus we will have tarnished our original solution! So once more we must add new cells to $A^1_{(k)}, \ldots$

From this description it is hoped that the reader imagines that the only way to defeat this feedback loop is to embrace it. That is, if we have already performed the round-trip of adding cells to $A^1_{(k)}$, then to $A^0_{(k)}$, then to $A^1_{(k)}$, and so on, infinitely many times, the situation should stabilise. Indeed, as we seek a left adjoint we wish to compute the least fixed-point of this process.

Here now we may see the role for the subscript $(k)$ we have included: we will use it as an indicator of our progress in stepping through this process of attaching solutions. Our overall goal for each step $f_{(k)} \mapsto f_{(k+1)}$ is to produce a new object and an inclusion $f_{(k)} \rightarrowtail f_{(k+1)}$ such that all lifting problems against $Pf_{(k)}$ as displayed below-left, when post-composed by the inclusion $f_{(k)} \rightarrowtail f_{(k+1)}$ as displayed below-right, will admit solutions instead against $Pf_{(k+1)}$ as indicated.

$$
\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & A^1_{(k)} \arrow[d, "Pf_{(k)}"] \\
2_n \arrow[r, "\Phi"'] & C^1 \underset{TC^0}{\times} TA^0_{(k)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & A^1_{(k)} \arrow[r, tail] & A^1_{(k+1)} \arrow[d, "Pf_{(k+1)}"] \\
2_n \arrow[r, "\Phi"'] \arrow[urr, dashed] & C^1 \underset{TC^0}{\times} TA^0_{(k)} \arrow[r, tail] & C^1 \underset{TC^0}{\times} TA^0_{(k+1)}
\end{tikzcd}
$$

By our above analysis of the feedback loop, inescapably we do not promise to solve all lifting problems at any finite step $f_{(k)}$, but only that all of the problems faced by any finite step are solved at the next. Although we may yet introduce additional, unsolved problems at each finite next step, it turns out that the fixed-point is immune from such issues. Let us suggest a not-entirely-accurate-but-nevertheless-useful intuition for phenomenon: any lifting problem against the fixed-point is solved at the next step of the fixed-point, viz., the fixed-point itself. Readers wishing to know why this is not *precisely* accurate are invited to read on![10]

To that end, let us make rigorous our ideas and begin by identifying the core process, $f_{(k)} \mapsto f_{(k)}$, whose fixed-point we seek.

**The successor $T$-graph morphism** Our task is as follows: given an object of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ as in (A.2.1.a), we wish to construct $f_{(k+1)} \colon A_{(k+1)} \to C$ a morphism of $T$-graphs along with the data of lifts to satisfy $f_{(k+1)} \in \mathrm{ob}(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ compatible with an inclusion $f_{(k)} \rightarrowtail f_{(k+1)}$ in that same category.

We shall achieve this process in several steps, and the order of dependencies is as follows.

1. Given an object $f_{(k)} \in \mathrm{ob}(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ we define the the successor globular sets $A^0_{(k+1)}$ and $A^1_{(k+1)}$ (Definition A.2.5).
2. For these successor globular sets we define the successor $C$-fibre morphisms, $f^{0,1}_{(k+1)} \colon A^{0,1}_{(k+1)} \to C$ (Definition A.2.7).
3. We define two maps of utility that will aide us in the next step (Definition A.2.8).
4. Using all of the above, we give the following interdependent definitions.
   (a) From the ancestor $f_{(k)} \colon A_{(k)} \to C$ we define the so-called coherent inclusion[11] of globular sets $c^{0,1}_{(k)} \colon A^{0,1}_{(k)} \to A^{0,1}_{(k+1)}$.
   (b) From the coherent inclusion $c^{0,1}_{(k)} \colon A^{0,1}_{(k)} \to A^{0,1}_{(k+1)}$ we define the successor $T$-graph over $C$, $f_{(k+1)} \colon A_{(k+1)} \to C$.

   Both of these definitions are split as a base case (Lemma A.2.10) and an inductive step (Lemma A.2.12).

[10] or to look at the footnote: the missing ingredient is that every lifting problem against the fixed-point is in fact an inclusion of a lifting problem against some finite stage, and hence solved at the next step and therefore solved in the fixed-point.

[11] The rigorous sense of coherence we defer to Lemma A.2.20 later, and so for the time being we ask the reader to extend their generosity in assessing our choice of nomenclature.

5. Finally we prove Lemma A.2.15 that the successor $f_{(k+1)}: A_{(k+1)} \to C$ is an object of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$, and that the coherent inclusions are a map $c_{(k)}: f_{(k)} \to f_{(k+1)}$ in that category.

Once we have all of these definitions in place, we will prove various nice properties about the successor and ultimately leverage it to give the left adjoint we seek.

Going into these definitions it will prove advantageous to introduce some notation for the truncation of a globular set.

*Notation* A.2.2. Given a globular set $X \in \mathrm{ob}\,\widehat{\mathbb{G}}$ let us write $\overline{X}^m$ to mean the globular set that agrees with $X$ in dimensions $m' \leq m$ and otherwise empty in dimensions $m' > m$. We will extend this to $m = -1$, $\overline{X}^{(-1)}$, to mean the empty globular set. ◁

As promised we begin with the successor globular sets, and their dependencies. First we will have need of globular sets which record the lifting problems faced by $Pf_{(k)}$. For this purpose we will have two such: $L^0(f_{(k)})$ and $L^1(f_{(k)})$. Both of these have as $n$-cells the lifting problems faced by $Pf_{(k)}$, but position them differently over their lower cells, which are respectively those of $A^0_{(k)}$ and $A^1_{(k)}$. In the first case, we say that a lifting problem—that is, an $n$-cell of $L^0(f_{(k)})$—has as its source and target in $A^0_{(k)}$ the value of its underlying boundary component $\partial_n \rightarrowtail A^1_{(k)}$ post-composed by $A^\ell_{(k)}$—the source and target a solution of the problem *would* have. In the second case, every lifting problem—that is, an $n$-cell of $L^1(f_{(k)})$—already has a well-defined boundary in $A^1_{(k)}$, so we simply use this—again, the source and target a solution of the problem would have. Let us now make this rigorous.

**Definition A.2.3.** For fixed $n \in \mathbb{N}$, and morphism $f_{(k)}: A_{(k)} \to C$ define the set $L(f_{(k)})$ of **$n$-dimensional lifting problems for $f_{(k)}$** to be the vertex of the below pullback in Set where $C^1 \times_{TC^0} TA^0_{(k)}$ and $Pf_{(k)}$ arise from the pullback (A.2.1.b).

$$\begin{tikzcd}
L(f_{(k)}) \arrow[rr, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \widehat{\mathbb{G}}\left(\partial_n, A^1_{(k)}\right) \arrow[d, "(Pf_{(k)})_*"] \\
\widehat{\mathbb{G}}\left(2_n, C^1 \times_{TC^0} TA^0_{(k)}\right) \arrow[rr, "\text{restrict}"'] & & \widehat{\mathbb{G}}\left(\partial_n, C^1 \times_{TC^0} TA^0_{(k)}\right)
\end{tikzcd}$$

From this set $L(f_{(k)})$ we define two globular sets $L^0(f_{(k)})$ and $L^1(f_{(k)})$ in dimensions $n$ and lower as follows, with higher dimensions taken empty.

$$L^1(f_{(k)}) :\equiv \left[ L(f_{(k)}) \xrightarrow{p} \widehat{\mathbb{G}}(\partial_n, A^1_{(k)}) \rightrightarrows \widehat{\mathbb{G}}(2_{n-1}, A^1_{(k)}) \overset{y}{\cong} (A^1_{(k)})_{n-1} \rightrightarrows \cdots \right]$$

$$L^0(f_{(k)}) :\equiv \left[ L(f_{(k)}) \xrightarrow{(A^\ell_{(k)})_* p} \widehat{\mathbb{G}}(\partial_n, A^0_{(k)}) \rightrightarrows \widehat{\mathbb{G}}(2_{n-1}, A^0_{(k)}) \overset{y}{\cong} (A^0_{(k)})_{n-1} \rightrightarrows \cdots \right]$$

It may be verified that both $L^0(f_{(k)})$ and $L^1(f_{(k)})$ are the on-objects part of the **lifting problem functors** $L^0, L^1: T\text{-Grph}/C \to \widehat{\mathbb{G}}$. If $h = (h^1, h^0): f_{(k)} \to g$ is a morphism of $T$-Grph$/C$, then we define a function $L(h)$ by the following pullback factorisation.

$$\begin{tikzcd}
L(f_{(k)}) \arrow[dr, dashed, "L(h)"] \arrow[drr, bend left, "(h^1)_* p_{f_{(k)}}"] \arrow[ddr, bend right, "(C^1 \times_{TC^0} Th^0)_* q_{f_{(k)}}"'] & & \\
& L(g) \arrow[r, "p_g"] \arrow[d, "q_g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\mathbb{G}}\Big(\partial_n, A^1\Big) \arrow[d, "(Pg)_*"] \\
& \widehat{\mathbb{G}}\Big(2_n, C^1 \times_{TC^0} TA^0\Big) \arrow[r, "\text{restrict}"'] & \widehat{\mathbb{G}}\Big(\partial_n, C^1 \times_{TC^0} TA^0\Big)
\end{tikzcd}$$

With this, we define $L^0(h)$ and $L^1(h)$ to the globular maps acting in dimension $n$ by $L(h)$, and in lower dimensions by $h^1$ and $h^0$ respectively. ◁

*Remark* A.2.4. The function $L(h)$ may be seen to send a lifting-problem, a morphism $(\partial\varphi, \Phi)\colon (\partial_n \rightarrowtail 2_n) \to P(f_{(k)})$ of $\operatorname{Ar}\widehat{\mathbb{G}}$, to the lifting problem against $Pg$ obtained by post-composition with $Ph$ in $\operatorname{Ar}\widehat{\mathbb{G}}$. The functors $L^0$ and $L^1$ merely extend this action to book-keeping relating to the source and target in lower dimensions. ◁

We are now ready to give the successor globular sets in terms of the globular sets of lifting problems just defined. The successor globular sets $A^{0,1}_{(k+1)}$ agree with their ancestors $A^{0,1}_{(k)}$ in every dimension but $n$. In that dimension, they both have as additional $n$-cells all of the lifting problems of $\partial_n \rightarrowtail 2_n$ faced by $Pf_{(k)}$. Importantly, there are no higher cells in $A^1_{(k+1)}$ which have as source or target any of the new cells of $L(f_{(k)})$. In this way, we have formally added solutions to all lifting problems faced by $Pf_{(k)}$ to $A^0_{(k+1)}$ and $A^1_{(k+1)}$ is some compatible way. The precise sense in which this is true is addressed in Lemma A.2.18.

**Definition A.2.5.** The **successor globular sets** $A^0_{(k+1)}$ **and** $A^1_{(k+1)}$ are defined to be the following pushouts in $\widehat{\mathbb{G}}$.

$$\begin{tikzcd}
\overline{A^0_{(k)}}^{n-1} \arrow[r, tail] \arrow[d, tail] & A^0_{(k)} \arrow[d, tail, "\iota^0_A"] \\
L^0(f_{(k)}) \arrow[r, tail, "\iota^0_L"'] & A^0_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A^1_{(k)}}^{n-1} \arrow[r, tail] \arrow[d, tail] & A^1_{(k)} \arrow[d, tail, "\iota^1_A"] \\
L^1(f_{(k)}) \arrow[r, tail, "\iota^1_L"'] & A^1_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

◁

Next we must equip the successor globular sets with morphisms to $C^0$ and $C^1$ as appropriate. Ultimately these $C$-fibre morphisms will serve the role of a morphism of $T$-graphs $A_{(k+1)} \to C$. Note that in order to give such a morphism we need only specify where the new cells of $A^{0,1}_{(k+1)}$ map in some compatible manner with $f_{(k)}$ and the structure of $A^{\ell,r}_{(k+1)}$. We will shortly dictate that the new cells of $A^1_{(k+1)}$—those $n$-dimensional lifting problems against $Pf_{(k)}$ formally appearing as $n$-cells—should map to their underlying cell of $C^1$. In the case of $A^0_{(k+1)}$, those same new cells map to $C^\ell$ of their underlying cell in $C^1$—that is, we consider the cells of $A^0_{(k+1)}$ to be formally *solved* lifting problems and so they should live over the appropriate fibre of $C^\ell$.

To make these definitions reasonably compact, let us introduce some notation for globular maps.

*Notation* A.2.6. To name a globular map $L^0(f) \to X$ or $L^1(f) \to X$ we shall find it convenient to write $[h,k]\colon L^i(f) \to X$ to mean:

1. implicitly there is a function $h\colon L(f) \to X_n$ and a globular map $k\colon \overline{A^i}^{n-1} \to X$,
2. the assignment $L^i(f) \to X$ given in dimensions as
   (a) trivial in high dimensions for $L^i(f)$ is empty,
   (b) $h$ sending $n$-cells of $L^i(f)$ to $n$-cells of $X$,
   (c) in lower dimensions $k$ sending $(n-m)$-cells of $L^i(f) = \overline{A^i}^{n-1}$ to cells of $X$,

   obeys the globular relations, and
3. the resulting globular map shall be called $[h,k]$.

◁

Now we are ready to define the successor $C$-fibre morphisms $f^{0,1}_{(k+1)}$.

**Definition A.2.7.** The **successor $C$-fibre morphisms** are the pair of globular morphisms $(f^0_{(k+1)}\colon A^0_{(k+1)} \to C,\ f^1_{(k+1)}\colon A^1_{(k+1)} \to C)$ induced as below by the universal property of the pushout

$$
\begin{tikzcd}
\overline{A^0_{(k)}}^{n-1} \arrow[r, tail] \arrow[d, tail] & A^0_{(k)} \arrow[d, tail, "\iota^0_A"] \arrow[ddr, bend left, "f^0_{(k)}"] & \\
L^0(f_{(k)}) \arrow[r, tail, "\iota^0_L"'] \arrow[drr, bend right, "{[C^\ell \pi_C y q, f^0_{(k)}]}"'] & A^0_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f^0_{(k+1)}"'] & \\
 & & C^0
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A^1_{(k)}}^{n-1} \arrow[r, tail] \arrow[d, tail] & A^1_{(k)} \arrow[d, tail, "\iota^1_A"] \arrow[ddr, bend left, "f^1_{(k)}"] & \\
L^1(f_{(k)}) \arrow[r, tail, "\iota^1_L"'] \arrow[drr, bend right, "{[\pi_C y q, f^1_{(k)}]}"'] & A^1_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f^1_{(k+1)}"'] & \\
 & & C^1
\end{tikzcd},
$$

where the maps labelled $[C^\ell \pi_C y q, f^0_k]$ and $[\pi_C y q, f^1_{(k)}]$ are written in the convention of Notation A.2.6. ◁

The next pieces of supporting machinery are specific globular maps out of the globular sets of lifting problems which we shall call $L^\ell(f_{(k)})\colon L^1(f_{(k)}) \to L^0(f_{(k)})$ and $L^r(f_{(k)})\colon L^1(f_{(k)}) \to TA^0_{(k)}$. The former map, will be used to understand how the newly present cells of $A^1_{(k+1)}$ should be mapped to $A^0_{(k+1)}$, while $L^r(f_{(k)})$ will serve the analogous role but for $TA^0_{(k+1)}$ instead. In the former case we wish to view the newly attached cells as living over their exact copies in $A^0_{(k+1)}$, and in the latter case we wish to project out from these new cells to their underlying pastings in $TA^0_{(k)}$.

**Definition A.2.8.** In the style of Notation A.2.6, let us define the **lifting problem codomain and domain maps** respectively as follows.

$$
\begin{aligned}
L^\ell(f_{(k)}) &:\equiv [\mathrm{id}, A^\ell_{(k)}]\colon L^1(f_{(k)}) \to L^0(f_{(k)}) \\
L^r(f_{(k)}) &:\equiv [\pi_A y q, A^r_{(k)}]\colon L^1(f_{(k)}) \to TA^0_{(k)}
\end{aligned}
$$

◁

*Remark* A.2.9. Note that the implicit claim that globularity is true of the lifting problem domain map is witnessed by the following commutative diagram.

$$
\begin{tikzcd}
L(f_{(k)}) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\mathbb{G}}\left(2_n, C^1 \times_{TC^0} TA^0_{(k)}\right) \arrow[r, phantom, "\overset{y}{\cong}"] \arrow[d, "\text{restrict}"] & \left(C^1 \times_{TC^0} TA^0_{(k)}\right)_n \arrow[r, "\pi_A"] & (TA^0_{(k)})_n \arrow[ddd, shift left] \arrow[ddd, shift right] \\
\widehat{\mathbb{G}}\left(\partial_n, A^1_{(k)}\right) \arrow[r, "(Pf_{(k)})_*"] \arrow[d, shift left] \arrow[d, shift right] & \widehat{\mathbb{G}}\left(\partial_n, C^1 \times_{TC^0} TA^0_{(k)}\right) \arrow[d, shift left] \arrow[d, shift right] & & \\
\widehat{\mathbb{G}}\left(2_{n-1}, A^1_{(k)}\right) \arrow[d, "y"', "\cong"] & \widehat{\mathbb{G}}\left(2_{n-1}, C^1 \times_{TC^0} TA^0_{(k)}\right) \arrow[d, "y", "\cong"'] & & \\
(A^1_{(k)})_{n-1} \arrow[r, "Pf_{(k)}"] \arrow[rrr, bend right, "A^r_{(k)}"'] \arrow[d, shift left] \arrow[d, shift right] & (C^1 \times_{TC^0} TA^0_{(k)})_{n-1} \arrow[rr, "\pi_A"] & & (TA^0_{(k)})_{n-1} \arrow[d, shift left] \arrow[d, shift right] \\
\vdots & & & \vdots
\end{tikzcd}
$$

◁

With these maps we are ready to define the successor $T$-graph.

Here for the first time we will give attention to the subscript $(k)$ appearing throughout this section. So far we have treated it as an arbitrary label and used it as a convenient indication of succession, but now we predicate our constructions on the positivity of the index appearing in the subscript. This is a foreshadowing of the subtle difference in applying the overall successor construction "for the first time" as opposed to "iteratively". In the former case we will be disregarding all information about extant solutions to $n$-dimensional lifting problems which the object of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ might happen to have, but in the latter case we will need to carefully preserve our work.

The following two lemmas contain what amounts to an inductive definition of successor $T$-graphs over $C$, and coherent inclusions between them. However we have chosen to split this up into a base case and an inductive step, for these constructions involve some (inductive) assumptions on the commutativity of certain diagrams. Owing to their apparent density, in the below lemmas we have underlined all of the claims we will need to prove as a reading aid.

**Lemma A.2.10.** *Fix an object $f_{(0)} \in \mathrm{ob}(T\text{-Grph}/C)$.*

*The **first coherent inclusions**, $c^0_{(1)} : A^0_{(0)} \to A^0_{(1)}$ and $c^1_{(0)} : A^1_{(0)} \to A^1_{(1)}$, are defined to be the maps $c^0_{(0)} :\equiv \iota^0_A$ and $c^1_{(0)} :\equiv \iota^1_A$ of Definition A.2.5.*

*The **first successor $T$-graph over $C$** is defined to be the diagram*

$$
\begin{tikzcd}
A^0_{(1)} \arrow[d, "f^0_{(1)}"'] & A^1_{(1)} \arrow[l, "A^\ell_{(1)}"'] \arrow[r, "A^r_{(1)}"] \arrow[d, "f^1_{(1)}"] & TA^0_{(1)} \arrow[d, "Tf^0_{(1)}"] \\
C^0 & C^1 \arrow[l, "C^\ell"] \arrow[r, "C^r"'] & TC^0
\end{tikzcd}
\tag{A.2.11}
$$

*where the maps $f^{0,1}_{(1)}$ are the successor $C$-fibre maps of Definition A.2.7, and the legs $A^{r,\ell}_{(1)}$ arise from the universal property of pushouts as follows.*

$$\begin{tikzcd}
\overline{A^1_{(k)}}^{n-1} \arrow[r, tail] \arrow[dd, tail] & A^1_{(k)} \arrow[dr, bend left, "A^\ell_{(0)}"] \arrow[dd, tail, "\iota^1_A"] & \\
& & A^0_{(0)} \arrow[dd, "\iota^0_A"] \\
L^1(f_{(k)}) \arrow[r, tail, "\iota^1_L"'] \arrow[dr, bend right, "L^\ell(f_{(0)})"'] & A^1_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "A^\ell_{(1)}"'] & \\
& L^0(f_{(0)}) \arrow[r, "\iota^0_L"'] & A^0_{(1)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A^1_{(k)}}^{n-1} \arrow[r, tail] \arrow[dd, tail] & A^1_{(k)} \arrow[dr, bend left, "A^r_{(0)}"] \arrow[dd, tail, "\iota^1_A"] & \\
& & TA^0_{(0)} \arrow[dd, "T\iota^0_A"] \\
L^1(f_{(k)}) \arrow[r, tail, "\iota^1_L"'] \arrow[dr, bend right, "L^r(f_{(0)})"'] & A^1_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "A^r_{(1)}"'] & \\
& TA^0_{(0)} \arrow[r, "Tc^0_{(0)}"'] & TA^0_{(1)}
\end{tikzcd}$$

*We <u>claim</u> that* (A.2.11) *is commutative, and we <u>claim</u> that $c_{(0)} = (c^0_{(0)}, c^1_{(0)})$ is a monomorphism $c_{(0)} \colon f_{(0)} \to f_1$ of $T$-*Grph$/C$ *which is the identity in all dimensions but $n$.*

*Proof.* That (A.2.11) is commutative may be deduced by universal property: by pre-composing both squares with the defining inclusions $A^1_{(0)} \rightarrowtail A^1_{(1)} \leftarrowtail L^1(f_{(0)})$. Pre-Composition by inclusion $A^1_{(0)} \rightarrowtail A^1_{(1)}$ immediately shows the relevant equations hold because $f_{(0)}$ is in particular a morphism of $T$-graphs. The verification of the equations under pre-composition by the inclusion $\iota^1_L$ is not substantially more involved, so we suppress the details here. Thus $f_{(1)}$ is an object of $T$-Grph$/C$. By unwinding definitions it is immediate that $c_{(0)}$ is a morphism of $T$-Grph$/C$, and because each component is monic and the identity in all dimensions but $n$, $c_0$ enjoys the same properties. ∎

**Lemma A.2.12.** *Fix $k \in \mathbb{N}$, and suppose that $f_{(k)}$ is an object of $T$-*Grph$/C$*. Suppose that there are monomorphisms $c^{0,1}_{(k)} \colon A^{0,1}_{(k)} \rightarrowtail A^{0,1}_{(k+1)}$ which are the identity in all dimensions but $n$. From these monomorphisms we construct the maps $A^\ell_{(k+1)} \colon A^1_{(k+1)} \to A^0_{(k+1)}$ and $A^r_{(k+1)} \colon A^1_{(k+1)} \to TA^0_{(k+1)}$ as in* (A.2.13) *and* (A.2.14) *below (where all subscripts are reduced by one). Suppose finally that the result is an object $f_{(k+1)} \in$ *ob$(T$-Grph$/C)$ *and that $c_{(k)} = (c^0_{(k)}, c^1_{(k)}) \colon f_{(k)} \to f_{(k+1)}$ is a monomorphism.*

*We define the **subsequent coherent inclusion** to be the pair of maps*

$$(c^0_{(k+1)} \colon A^0_{(k+1)} \to A^0_{(k+2)},\ c^1_{(k+1)} \colon A^1_{(k+1)} \to A^1_{(k+2)})$$

*arising from the following universal properties of the pushout.*

$$\begin{tikzcd}
\overline{A^0_{(k)}}^{n-1} \arrow[r, tail] \arrow[dd, tail] & A^0_{(k)} \arrow[dr, tail, bend left, "\iota^0_A"] \arrow[dd, tail, "\iota^0_A"] & \\
& & A^0_{(k+1)} \arrow[dd, tail, "\iota^0_A"] \\
L^0(f_{(k)}) \arrow[r, tail, "\iota^0_L"'] \arrow[dr, tail, bend right, "L^0(c_{(k)})"'] & A^0_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, tail, dashed, "c^0_{(k+1)}"'] & \\
& L^0(f_{(k+1)}) \arrow[r, tail, "\iota^0_L"'] & A^0_{(k+2)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A^1_{(k)}}^{n-1} \arrow[r, tail] \arrow[dd, tail] & A^1_{(k)} \arrow[dr, tail, bend left, "\iota^1_A"] \arrow[dd, tail, "\iota^1_A"] & \\
& & A^1_{(k+1)} \arrow[dd, tail, "\iota^1_A"] \\
L^1(f_{(k)}) \arrow[r, tail, "\iota^1_L"'] \arrow[dr, tail, bend right, "L^1(c_k)"'] & A^1_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, tail, dashed, "c^1_{(k+1)}"'] & \\
& L^1(f_{(k+1)}) \arrow[r, tail, "\iota^1_L"'] & A^1_{(k+2)}
\end{tikzcd}$$

*We define **subsequent successor $T$-graph over $C$** to be the diagram*

$$
\begin{tikzcd}
A^0_{(k+2)} \arrow[d, "f^0_{(k+2)}"'] & A^1_{(k+2)} \arrow[l, "A^\ell_{(k+2)}"'] \arrow[r, "A^r_{(k+2)}"] \arrow[d, "f^1_{(k+2)}"] & TA^0_{(k+2)} \arrow[d, "Tf^0_{(k+2)}"] \\
C^0 & C^1 \arrow[l, "C^\ell"] \arrow[r, "C^r"'] & TC^0
\end{tikzcd}
\tag{A.2.13}
$$

,

*where the maps $f^{0,1}_{(k+2)}$ are the successor $C$-fibre maps of Definition A.2.7, and the legs $A^{\ell,r}_{(k+2)}$ arise from the universal property of pushouts as follows.*

$$
\begin{tikzcd}
\overline{A^1_{(k+1)}}^{n-1} \arrow[r, tail] \arrow[dd, tail] & A^1_{(k+1)} \arrow[dr, bend left, "A^\ell_{(k+1)}"] \arrow[dd, tail, "\iota^1_A"] & \\
& & A^0_{(k+1)} \arrow[dd, "\iota^0_A"] \\
L^1(f_{(k+1)}) \arrow[r, tail, "\iota^1_L"'] \arrow[dr, bend right, "L^\ell(f_{(k+1)})"'] & A^1_{(k+2)} \arrow[dr, dashed, "A^\ell_{(k+2)}"'] & \\
& L^0(f_{(k+1)}) \arrow[r, "\iota^0_L"'] & A^0_{(k+2)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A^1_{(k+1)}}^{n-1} \arrow[r, tail] \arrow[dd, tail] & A^1_{(k+1)} \arrow[dr, bend left, "A^r_{(k+1)}"] \arrow[dd, tail, "\iota^1_A"] & \\
& & TA^0_{(k+1)} \arrow[dd, "T\iota^0_A"] \\
L^1(f_{(k+1)}) \arrow[r, tail, "\iota^1_L"'] \arrow[dr, bend right, "L^r(f_{(k+1)})"'] & A^1_{(k+2)} \arrow[dr, dashed, "A^r_{(k+2)}"'] & \\
& TA^0_{(k+1)} \arrow[r, "Tc^0_{(k+1)}"'] & TA^0_{(k+2)}
\end{tikzcd}
\tag{A.2.14}
$$

*We <u>claim</u> that* (A.2.13) *is commutative, and we <u>claim</u> that $c_{(k+1)} = (c^0_{(k+1)}, c^1_{(k+1)})$ is a monomorphism $c_{(k+1)} : f_{(k+1)} \to f_{(k+2)}$ of $T$-$\mathrm{Grph}/C$ which is the identity in all dimensions but $n$.*

*Proof.* Unfortunately for the most part this proof is uninteresting busywork. All of the claims, save the last about monic components which are the identity in dimensions other than $n$, amount to verifying that the following dozen (!) equations hold.

$$
\begin{aligned}
f^0_{(k+2)} A^\ell_{(k+2)} \iota^1_A &= C^\ell f^1_{(k+2)} \iota^1_A &\quad \text{and} \quad && f^0_{(k+2)} A^\ell_{(k+2)} \iota^1_L &= C^\ell f^1_{(k+2)} \iota^1_L \ , \\
(Tf^0_{(k+2)}) A^r_{(k+2)} \iota^1_A &= C^r f^1_{(k+2)} \iota^1_A &\quad \text{and} \quad && (Tf^0_{(k+2)}) A^r_{(k+2)} \iota^1_L &= C^r f^1_{(k+2)} \iota^1_L \ , \\
f^0_{(k+2)} c^0_{(k+1)} \iota^0_A &= f^0_{(k+1)} \iota^0_A &\quad \text{and} \quad && f^0_{(k+2)} c^0_{(k+1)} \iota^0_L &= f^0_{(k+1)} \iota^0_L \ , \\
f^1_{(k+2)} c^1_{(k+1)} \iota^1_A &= f^1_{(k+1)} \iota^1_A &\quad \text{and} \quad && f^1_{(k+2)} c^1_{(k+1)} \iota^1_L &= f^1_{(k+1)} \iota^1_L \ , \\
A^\ell_{(k+2)} c^1_{(k+1)} \iota^1_A &= c^0_{(k+2)} A^\ell_{(k+1)} \iota^1_A &\quad \text{and} \quad && A^\ell_{(k+2)} c^1_{(k+1)} \iota^1_L &= c^0_{(k+2)} A^\ell_{(k+1)} \iota^1_L \ , \\
A^r_{(k+2)} c^1_{(k+1)} \iota^1_A &= (Tc^0_{(k+2)}) A^r_{(k+1)} \iota^1_A &\quad \text{and} \quad && A^r_{(k+2)} c^1_{(k+1)} \iota^1_L &= (Tc^0_{(k+2)}) A^r_{(k+1)} \iota^1_L
\end{aligned}
$$

The entire left-hand column, however, involves nothing more than expanding definitions and occasional appeal to surface facets of the inductive hypothesis and so we omit it. Of the remaining six, the first two, indexed down the right-hand column, involve only light considerations of the inductive hypothesis and so we omit these as well. Of those which remain, the first is proven entirely similarly to the second, and so we omit it.

Consider the desired equality $f^1_{(k+2)} c^1_{(k+1)} \iota^1_L = f^1_{(k+1)} \iota^1_L$, and let us expand definitions to compute that the left-hand side is $[\pi_C yq, f^1_{(k+1)}] L^1(c_{(k)})$ while the right-hand side is $[\pi_C yq, f^1_{(k)}]$. Our inductive hypothesis informs us that $c_{(k)}$ is the identity in dimensions other than $n$, so by examining Remark A.2.4 we see that $L^1(c_{(k)})$ is the identity in dimensions less than $n$. Furthermore, $f^1_{(k+1)}$ by definition agrees with $f^1_{(k)}$

in dimensions less than $n$ and so overall the equality is true on all cells in dimensions less than $n$. On $n$ cells, that is, lifting problems against $Pf_{(k)}$, the left-hand side first post-composes by $Pc_{(k)}$ before projecting to $C^1$, while the right-hand side simply projects to $C^1$. However, these operations agree by the definition of $Pc_{(k)}$. Ergo, the equality stands in all dimensions.

Now let us examine $A^{\ell}_{(k+2)}c^1_{(k+1)}\iota^1_L$ and $c^0_{(k+2)}A^{\ell}_{(k+1)}\iota^1_L$. The former composite may be simplified to $\iota^0_L L^{\ell}(f_{(k+1)})L^1(c_{(k)})$ while the latter is reducible to $\iota^0_L L^0(c_{(k)})L^{\ell}(f_{(k)})$. Let us expand $L^{\ell}$ of Definition A.2.8, and we shall argue that the earlier equality $[\mathrm{id}, A^{\ell}_{(k+1)}]L^1(c_{(k)}) = L^0(c_{(k)})[\mathrm{id}, A^{\ell}_{(k)}]$ is obtained in every dimension. In dimensions less than $n$, we may inspect the definition of $A^{\ell}_{(k+1)}$ to see that it agrees with $A^{\ell}_{(k)}$ and furthermore—as argued above—$L^{0,1}(c_{(k)})$ is the identity in these dimensions too. Thus the equality is obtained in dimensions less than $n$. On $n$-cells, once more appealing to Remark A.2.4, we see that both $L^{0,1}(c_{(k)})$ are given by post-composing lifting problems with $Pc_{(k)}$. Ergo, the equality stands in all dimensions.

The last comparison to be made is between

$$A^r_{(k+2)}c^1_{(k+1)}\iota^1_L = (Tc^0_{(k+1)})[\pi_A yq, A^r_{(k+1)}]L^1(c_{(k)})$$

and

$$(Tc^0_{(k+1)})A^r_{(k+1)}\iota^1_L = (Tc^0_{(k+1)})(Tc^0_{(k)})[\pi_A yq, A^r_{(k)}] \ .$$

Again our strategy will be to argue by dimension, and show that the earlier equality before post-composition by $Tc^0_{(k+1)}$ is obtained. On cells of dimension less than $n$, by our arguments above, both assignments are given by $A^r_{(k)}$ and hence agree. On $n$-dimensional cells, that is lifting problems as below, the equality is obtained by the commutativity of the rendered diagram.

$$\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & A^1_{(k)} \arrow[r, "c^1_{(k)}"] \arrow[d, dashed, "Pf_{(k)}"] & A^1_{(k+1)} \arrow[d, "Pf_{(k+1)}"] \\
2_n \arrow[r, "\Phi"'] & C^1 \times_{TC^0} TA^0_{(k)} \arrow[r, "C^1 \times_{TC^0} Tc^0_{(k)}"'] \arrow[d, "\text{proj.}"'] & C^1 \times_{TC^0} TA^0_{(k+1)} \arrow[d, "\text{proj.}"] \\
 & TA^0_{(k)} \arrow[r, "Tc^0_{(k)}"'] & TA^0_{(k+1)}
\end{tikzcd}$$

Now that we have established that $f_{(k+1)}$ is an object of $T$-Grph$/C$ and that $c_{(k+1)}$ is a morphism of that same category, the outstanding claims are that $c_{(k+1)}$ is the identity in dimensions other than $n$ and that it is monic. The former property is apparent from inspecting the defining pullbacks and the same property of $c_{(k)}$, and so $L^{0,1}(c_{(k)})$. The latter follows from the fact that monomorphisms are expressible as pullbacks, and so a cartesian monad preserves monomorphisms. Furthermore monomorphisms are stable under pullbacks, so that overall by examining the inductive definition each $c_{(k+1)}$ is a monomorphism. ■

At this point we have inductively given, for an object $f_{(0)} \in \mathrm{ob}(T\text{-Grph}/C)$, a family of objects $\{f_{(k)}\}_{k\in\mathbb{N}}$ of the same category as well as a family of inclusions $\{c_{(k)} \colon f_{(k)} \rightarrowtail f_{(k+1)}\}$. However, we are interested not in $T$-graphs fibred over $C$, but in objects of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$. Nevertheless, all of the work we have done so far may readily be realised in this more structured context.

**Lemma A.2.15.** *Given $f_{(0)} \in \mathrm{ob}(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$, for each $k \in \mathbb{N}$ the $k^{th}$ successor $T$-graph over $C$, $f_{(k)}$, is once more an object of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ and the coherent inclusions extend to a monomorphism $c_{(k)} \colon f_{(k)} \to f_{(k+1)}$ in that same category.*

*Proof.* We must establish that each $f_{(k)}$ is supplied with the data of lifts against problems of dimensions less than $n$, and that each $c_{(k)}$ descends to a morphism of this data. However this is straightforward, all of the data of $f_{(k)}$ agrees with that of $f_{(0)}$ except in dimension $n$. In particular then, whatever lower dimension solutions $f_{(0)}$ had are still valid solutions for each $f_{(k)}$. Moreover, each $c_{(k)}$ is in particular the identity in dimensions less than $n$, so evidently descends to a morphism of these solutions. ∎

As this is a text on categories, it will not surprise the reader at all to meet the claim that this long-winded assignment of successors is functorial.

**Lemma A.2.16.** *The assignment of objects $f_{(k)} \mapsto f_{(k+1)}$ extends to an endofunctor $S_n$ of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$.*

*Proof.* Given objects $f_{(k)} \colon A_{(k)} \to C$ and $g_{(k)} \colon B_{(k)} \to C$ as well as a morphism $h_{(k)} \colon f_{(k)} \to g_{(k)}$ of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$, the definition of $h_{(k+1)} \colon f_{(k+1)} \to g_{(k+1)}$ amounts to the data of the globular map $L(h_{(k)}) \colon L(f_{(k)}) \to L(g_{(k)})$ of Definition A.2.3, combined with the universal property of the defining pushouts to induce the appropriate globular maps $A^{0,1}_{(k+1)} \to B^{0,1}_{(k+1)}$. Inevitably for such assignments defined by universal properties, $S_n$ is functorial. ∎

**Lemma A.2.17.** *The pairs $(c^0_{(k)}, c^1_{(k)})$ of globular morphisms extend to a natural inclusion $c \colon \mathrm{id}_{(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}} \Rightarrow S_n$, which is the identity in all dimensions but $n$.*

*Proof.* Natural transformations which are point-wise monic are monic, so naturality is the only outstanding claim. Inevitably this is given by a protracted inductive argument proceeding by universal-property-style proofs. As we have not spelt out the details of the functor $S_n$, we will not show the routine computations involved here either. ∎

The utility of the coherent inclusions becomes apparent once we introduce the canonical solutions to lifting problems possessed by each successor.

**Lemma A.2.18.** *Given a commutative square as below left, there exists a canonical solution $\ulcorner \partial\varphi, \Phi \urcorner^k$ to the induced lifting problem $(Pc_{(k)})(\partial\varphi, \Phi)$ as below-right.*

$$
\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & A^1_{(k)} \arrow[d, "Pf_{(k)}"] \\
2_n \arrow[r, "\Phi"'] & C^1 \underset{TC^0}{\times} TA^0_{(k)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & A^1_{(k)} \arrow[r, tail, "(Pc_{(k)})^1"] & A^1_{(k+1)} \arrow[d, "Pf_{(k+1)}"] \\
2_n \arrow[r, "\Phi"'] \arrow[urr, dashed, "\ulcorner \partial\varphi, \Phi \urcorner^k"] & C^1 \underset{TC^0}{\times} TA^0_{(k)} \arrow[r, tail] & C^1 \underset{TC^0}{\times} TA^0_{(k+1)}
\end{tikzcd}
\tag{A.2.19}
$$

*Proof.* The entire lifting problem above-left names the element $\langle \partial\varphi, \Phi \rangle \in L(f_{(k)})$, and so an $n$-cell

$$\ulcorner \partial\varphi, \Phi \urcorner^k :\equiv \iota_L^1 \langle \partial\varphi, \Phi \rangle \colon 2_n \to A^1_{(k+1)} \ .$$

By Definition A.2.3, this element is precisely over the $(n-1)$-dimensional boundary $(Pc_{(k)})^1 \partial\varphi = c^1_{(k)} \partial\varphi$ for $c_{(k)}$ is in particular the identity in dimension $n-1$. It remains to verify that the lower triangle is commutative.

Recall that $Pf_{(k+1)} = \langle f^1_{(k+1)}, A^r_{(k+1)} \rangle$, and by unravelling definitions we may evaluate the composites

$$f^1_{(k+1)} \ulcorner \partial\varphi, \Phi \urcorner^k = \pi_C \Phi \qquad \text{and} \qquad A^r_{(k+1)} \ulcorner \partial\varphi, \Phi \urcorner^k = (Tc^0_{(k)}) \pi_A \Phi \ ,$$

whence the lower triangle is commutative. ■

Moreover, the coherent inclusions are deserving of name.

**Lemma A.2.20.** *Given a lifting problem* $(\partial\varphi, \Phi)$ *for* $Pf_{(k)}$ *and the canonical solution* $\ulcorner \partial\varphi, \Phi \urcorner^k$ *for the induced lifting problem* $P(c_{(k)})(\partial\varphi, \Phi)$ *as in* (A.2.19)*, the coherent inclusion* $c_{(k+1)} \colon f_{(k+1)} \to f_{(k+2)}$ *has the property of preserving the canonical solution,*

$$(Pc_{(k+1)})^1 \ulcorner \partial\varphi, \Phi \urcorner^k = \ulcorner (Pc_{(k)})(\partial\varphi, \Phi) \urcorner^{k+1}$$

*Proof.* This is a matter of expanding definitions,

$$(Pc_{(k+1)})^1 \ulcorner \partial\varphi, \Phi \urcorner^k = c^1_{(k+1)} \iota^1_L \langle \partial\varphi, \Phi \rangle = \ulcorner L(c_{(k)}) \langle \partial\varphi, \Phi \rangle \urcorner^{k+1} = \ulcorner P(c_{(k)})(\partial\varphi, \Phi) \urcorner^{k+1} \ ,$$

where the last equality follows by Remark A.2.4. ■

Certainly there are other coherences between $f_{(k)}$ its arbitrary descendants, but given that we will be interested only in the linear diagrams indexed by $\omega$,

$$f_{(0)} \rightarrowtail f_{(1)} \rightarrowtail f_{(2)} \rightarrowtail \dots \ ,$$

the coherence we gave above generates all of the others for this shape.

Another pleasant property of these canonical solutions are preserved by the functorial action of $S_n$. The proof for the following lemma amounts a straightforward unwinding of definitions, combined with Remark A.2.4.

**Lemma A.2.21.** *Let* $h_{(k)} \colon f_{(k)} \to g_{(k)}$ *be a morphism in* $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$*. Then the successor morphism* $h_{(k+1)} \colon f_{(k+1)} \to g_{(k+1)}$ *preserves the canonical solutions to lifting problems against* $Pf_{(k)}$*. That is, for all lifting problems* $(\partial\varphi, \Phi)$ *against* $Pf_{(k)}$*, we have*

$$(Ph_{(k+1)})^1 \ulcorner \partial\varphi, \Phi \urcorner^{f_{(k)}} = \ulcorner P(h_{(k)})(\partial\varphi, \Phi) \urcorner^{h_{(k+1)}} \ . \qquad ■$$

Finally, the last important property of the successor $T$-graph morphism is that morphisms out of $f_{(k)} \in (T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ to objects in the image of the functor $V_n \colon (T\text{-Grph}/C)^{\mathrm{PRF}}_n \to (T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ factor uniquely through $f_{(k)} \rightarrowtail f_{(k+1)}$ when we restrict our attention to factorisations which appropriately preserve the canonical solutions present in $f_{(k+1)}$ to lifting problems against $f_{(k)}$. This property is essential to proving that the least fixed-point of $S_n$ has the correct universal property.

**Lemma A.2.22.** *Let $g \in \mathrm{ob}(T\text{-}\mathrm{Grph}/C)_n^{\mathrm{PRF}}$, and fix a morphism $h_{(0)}\colon f_{(0)} \to V_n g$ of $(T\text{-}\mathrm{Grph}/C)_{n-1}^{\mathrm{PRF}}$. There is a unique extension of $h_{(0)}$ to a family $\{h_{(k+1)}\colon f_{(k+1)} \to V_n g\}_{k\in\mathbb{N}}$ of morphisms which satisfy for all $k$:*

*(i)* $h_{(k+1)} c_{(k)} = h_{(k)}$, *and*

*(ii) for all lifting problems $(\partial\varphi, \Phi)$ as in* (A.2.19)*, the map $h_{(k+1)}$ preserves the canonical solution $\ulcorner \partial\varphi, \Phi \urcorner^k$ in the following sense.*

$$(Ph_{(k+1)})^1 \ulcorner \partial\varphi, \Phi \urcorner^k = \kappa_g((Ph_{(k)})(\partial\varphi, \Phi))$$

We shall see that conditions (i) and (ii) above essentially amount to the definition of each morphism $h_{(k+1)}$, so that uniqueness follows once we have settled existence. For clarity we have separated these arguments.

*Proof* (Existence). We shall give our construction inductively, noting that $h_{(0)}$ is already supplied, so suppose a morphism $h_{(k)}\colon f_{(k)} \to Vg$

Observe that the data of the split contraction $\kappa_g$ gives in particular a set function $\kappa_g\colon L(g) \to B^1_n$. Using these two functions we construct the globular maps $[B^\ell \kappa_g, \mathrm{id}]\colon L^0(g) \to B^0$ and $[\kappa_g, \mathrm{id}]\colon L^1(g) \to B^1$ (see Notation A.2.6). It is straightforward to check that these are indeed globular maps, and the proofs are similar so we produce only the commutative diagram for $[B^\ell \kappa_g, \mathrm{id}]$. The region marked $(\boxslash)$ is serially commutative by the definition of solutions to lifting problems.

$$\begin{tikzcd}
L(g) \arrow[rr, "\kappa_g"] \arrow[d, "p"'] & & B^1_n \arrow[r, "B^\ell"] \arrow[dd, shift left] \arrow[dd, shift right] & B^0_n \arrow[dd, shift left] \arrow[dd, shift right] \\
\widehat{\mathbb{G}}(\partial_n, B^1) \arrow[d, shift left] \arrow[d, shift right] & (\boxslash) & & \\
\widehat{\mathbb{G}}(2_{n-1}, B^1) \arrow[rr, "y"'] \arrow[d, "B^\ell y"'] & & B^1_{n-1} \arrow[r, "B^\ell"'] & B^0_{n-1} \arrow[d, equal] \\
(B^0_{(k)})_{n-1} \arrow[rrr, equal] & & & B^0_{n-1}
\end{tikzcd}$$

From these two globular maps we define the two components of $h_{(k+1)}$ via the universal property of pushouts.

$$\begin{tikzcd}
\overline{A^0_{(k)}}^{n-1} \arrow[r, tail] \arrow[d, tail] & A^0_{(k)} \arrow[d, tail, "\iota^0_A"] \arrow[ddr, bend left, "h^0_{(k)}"] & \\
L^0(f_{(k)}) \arrow[r, tail, "\iota^0_L"'] \arrow[drr, bend right, "{[B^\ell\kappa_g,\mathrm{id}]L^0(h_{(k)})}"'] & A^0_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "h^0_{(k+1)}"'] & \\
& & B^0
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A^1_{(k)}}^{n-1} \arrow[r, tail] \arrow[d, tail] & A^1_{(k)} \arrow[d, tail, "\iota^1_A"] \arrow[ddr, bend left, "h^1_{(k)}"] & \\
L^1(f_{(k)}) \arrow[r, tail, "\iota^1_L"'] \arrow[drr, bend right, "{[\kappa_g,\mathrm{id}]L^1(h_{(k)})}"'] & A^1_{(k+1)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "h^1_{(k+1)}"'] & \\
& & B^1
\end{tikzcd}$$

In order for the inductive construction to be valid, we need to prove that the pair $h_{(k+1)} = (h^0_{(k+1)}, h^1_{(k+1)})$ is a morphism $f_{(k+1)} \to Vg$. In fact we will (have to) prove simultaneously that $h_{(k)}$ is a morphism and that $h^{0,1}_{(k)} = h^{0,1}_{(k+1)} c^{0,1}_{(k)}$ holds serially. Once this is done, the induction will witness that each $h_{(k)}$ is a morphism and our pair of equalities will then be condition (i).

In the base case $h_{(0)}$ is a morphism by assumption and, by Lemma A.2.10, $c^0_{(0)} = \iota^0_A$ and $c^1_{(0)} = \iota^1_A$ so that $h^{0,1}_{(0)} = h^{0,1}_{(1)} c^{0,1}_{(0)}$ holds serially by definition.

For the inductive step, suppose that that $h_{(k)}$ is a morphism and $h^{0,1}_{(k)} = h^{0,1}_{(k+1)} c^{0,1}_{(k)}$ hold serially. Consider the following four equations whose joint truth witness the fact that $h_{(k+1)}$ is a morphism at least of the underlying $T$-graphs. That $h_{(k+1)}$ is then a morphism of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ is automatic: in dimensions other than $n$, $h^{0,1}_{(k+1)}$ are precisely the functions $h^{0,1}_{(k)}$ which preserve solutions to lifts against $k < n$ cells by assumption. Thus we have reduced the task of checking that $h_{(k+1)}$ is a morphism to the equations we now list.

$$\begin{aligned} f^1_{(k+1)} &= g^1 h^1_{(k+1)} & f^0_{(k+1)} &= g^0 h^0_{(k+1)} \\ B^\ell h^1_{(k+1)} &= h^0_{(k+1)} A^\ell_{(k+1)} & B^r h^1_{(k+1)} &= (T h^1_{(k+1)}) A^r_{(k+1)} \end{aligned} \tag{A.2.23}$$

Each of these equations may be determined to be valid by a universal property argument, that is, by inspecting their truth under the appropriate defining inclusions $A^{0,1}_{(k)} \rightarrowtail A^{0,1}_{(k+1)} \leftarrowtail L^{0,1}(f_{(k)})$. In the former case, under the inclusions of $A^{0,1}_{(k)}$ the equalities are all immediate. In the latter case some work is involved.

The second equation on the first line has an argument which is a small modification of the first-listed, so we show only the details for the latter. Consider then that $f^1_{(k+1)}\iota^1_L = [\pi_C y q, f^1_{(k)}]$ and that $g^1 h^1_{(k+1)} \iota^1_L = g^1[\kappa_g, \mathrm{id}]L^1(h_{(k)})$. In dimensions less than $n$, both of these functions are equal by our inductive hypothesis, $f^1_{(k)} = g^1 h^1_{(k)}$. In dimension $n$, that is on a lifting problem as exhibited in the below diagram, the first function traces the path $2_n \to C^1 \times_{TC^0} TA^0_{(k)} \to C^1$ while the latter may be seen to trace $2_n \to B^1 \to C^1 \times_{TC^0} TB^0 \to C^1$. Because the below diagram is commutative, these functions agree.

$$\begin{tikzcd}
\partial_n \arrow[r, "\partial\varphi"] \arrow[d, tail] & A^1_{(k)} \arrow[rr, "h^1_{(k)}"] \arrow[d, "Pf_{(k)}"'] & & B^1 \arrow[d, "Pg"] \\
2_n \arrow[r, "\Phi"'] \arrow[urrr, "\kappa_g\big((Ph_{(k)})(\partial\varphi,\Phi)\big)"] & C^1 \times_{TC^0} TA^0_{(k)} \arrow[rr, "C^1 \times_{TC^0} Th^0_{(k)}"'] \arrow[dr, "\text{proj.}"'] & & C^1 \times_{TC^0} TB^0 \arrow[dl, "\text{proj.}"] \\
& & C^1 &
\end{tikzcd}$$

The next equality, $B^\ell h^1_{(k+1)} \iota^1_L = h^0_{(k+1)} A^\ell_{(k+1)} \iota^1_L$ follows by a mechanical argument of unwrapping definitions and arguing in dimensions, and is less involved. Finally we wish to demonstrate $B^r h^1_{(k+1)} = (Th^1_{(k+1)}) A^r_{(k+1)}$. We may compute that, under pre-composition by $\iota^1_L$, the relevant terms to compare are $B^r[\kappa_g, \mathrm{id}]L^1(h_{(k)})$ and $(Th^0_{(k+1)})(Tc^0_{(k)})L^r(f_{(k)})$. Here we must use the other information contained in our inductive hypothesis, viz., $h^0_{(k)} = h^0_{(k+1)} c^0_{(k)}$. That is, it is enough to establish the equality $B^r[\kappa_g, \mathrm{id}]L^1(h_{(k)}) = (Th^0_{(k)})[\pi_A yq, A^r_{(k)}]$. Again we argue by dimensions to see that by hypothesis on $h_{(k)}$ this holds for cells in dimension less than $n$. On $n$-dimensional cells, that is, on lifting problems against $Pf_{(k)}$ as above, we may draw a similar commutative lifting diagram to that given above above and establish the claim.

To complete the induction it therefore remains to determine the validity the of claim that $h^{0,1}_{(k+1)} = h^{0,1}_{(k+2)} c^{0,1}_{(k+1)}$ serially. Again we argue under the defining inclusions of $A^{0,1}_{(k)}$ and $L^{0,1}(f_{(k)})$, and again in the former case the equalities are immediate. Thus to complete the induction we need only show that

$$h^0_{(k+1)} \iota^0_L = h^0_{(k+2)} c^0_{(k+1)} \iota^0_L \qquad \text{and} \qquad h^1_{(k+1)} \iota^1_L = h^1_{(k+2)} c^1_{(k+1)} \iota^1_L \ .$$

Fortunately, and in a pleasant change of affairs, these checks are straightforward. For instance, by the functoriality of $L^1$ and the assumption $h^1_{(k)} = h^1_{(k+1)} c^1_{(k)}$ we see that $h^1_{(k+2)} c^1_{(k+1)} \iota^1_L = [\kappa_g, \mathrm{id}]L^1(h^1_{(k+1)} c_{(k)}) = [\kappa_g, \mathrm{id}]L^1(h^1_{(k)}) = h^1_{(k+1)} \iota^1_L$ and similarly so for other equality.

At this point of the proof we have inductively constructed a family of morphisms $\{h_{(k)} \colon f_{(k)} \to Vg\}_{k \in \mathbb{N}}$ of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ which satisfy (i). To conclude the construction it remains then to validate (ii). We will now show this for each $k$ at once.

Note that every $n$-cell of $L^1(f_{(k)})$ is uniquely named by an element $\langle \partial\varphi, \Phi \rangle$ of $L(f_{(k)})$. By Remark A.2.4, $L(h_{(k+1)})$ evaluated at the $n$-cell $\langle \partial\varphi, \Phi \rangle$ is the $n$-cell $\langle (Ph_{(k)})(\partial\varphi, \Phi) \rangle$. Thus, because $(Ph_{(k+1)})^1 = h^1_{(k+1)}$ and $\ulcorner \partial\varphi, \Phi \urcorner^k = \iota^1_L \langle \partial\varphi, \Phi \rangle$, we see that (ii) is simply the constraint $h^1_{(k+1)} \iota^1_L = \kappa_g L(h_{(k)})$ on $n$-cells. Of course this was our very definition of $h^1_{(k+1)} \iota^1_L$ in dimension $n$, so (ii) is satisfied. ■

Now that we have shown existence, let us contend with uniqueness.

*Proof* (Uniqueness). We have just seen that (ii) uniquely determines each $h^1_{(k+1)}$ on the inclusion of $n$-cells of $L^1(f_{(k)})$ into $A^1_{(k+1)}$ to be $\kappa_g L(h_{(k)})$. To demonstrate uniqueness we must address the lower cells of $L^1(f_{(k)})$ and all of the cells of $A^1_{(k)} \rightarrowtail A^1_{(k+1)}$, and show that uniqueness is forced for $h^0_{(k)}$ too.

Let us therefore argue by induction. When $k = 0$ we see that $h^1_{(0)} = h^1_{(1)}\iota^1_A$ and $h^0_{(0)} = h^0_{(1)}\iota^0_A$ by (i) and the equality $c^1_{(0)} = \iota^1_A$. Thus $h^1_{(0)}$ is determined on all the cells of $A^1_{(0)}$, whose lower cells are the same as those of $L^1(f_{(0)})$, as well as the $n$-cells of $L^1(f_{(0)})$ by (ii). Ergo $h^1_{(1)}$ is uniquely determined on $A^1_{(0)}$. Similarly $h^0_{(1)}$ is determined on all of $A^0_{(0)}$, whose lower cells are the same as those of $L^0(f_{(0)})$, and the equation $B^\ell h^1_{(1)} = h^0_{(1)}A^\ell_{(1)}$ when constrained to $n$-cells shows that $h^0_{(1)} = B^\ell h^1_{(1)}$ for on $n$-cells the function $A^\ell_{(1)}$ is the identity.

Suppose now that $h^1_{(k+1)}$ and $h^0_{(k+1)}$ are uniquely determined, and consider $h^1_{(k+2)}$ and $h^0_{(k+2)}$. Although it may not be evident from the presentation of the globular sets $A^1_{(k+1)}$, in all dimensions $m \neq n$ we have $(A^1_{(k+1)})_m = (A^1_{(k+2)})_m$. From this we see that the only cells of $A^1_{(k+2)}$ outside of the image of $c^1_{(k+1)}$ are those lifting problems against $Pf_{(k+1)}$ which do not factor as a composite by $Pc_{(k)}$. Ergo, considering that we have $h^1_{(k+1)} = h^1_{(k+2)}c^1_{(k+1)}$ we see that $h^1_{(k+2)}$ is uniquely determined by $h^1_{(k+1)}$ on all the cells of $A^1_{(k+2)}$ except those lifting problems not factoring through $Pc_{(k)}$. On those problems, arguing as above, we see that (ii) uniquely determines $h^1_{(k+1)}$. Similarly, $h^0_{(k+2)}$ is uniquely determined by $h^0_{(k+1)}$ on all the cells of $A^0_{(k+2)}$ except those same novel lifting problems. As before, examining the equation $B^\ell h^1_{(k+2)} = h^0_{(k+2)}A^\ell_{(k+2)}$ in dimension $n$ shows that $h^0_{(k+2)}$ is determined on the novel lifting problems by $h^1_{(k+2)}$, and so uniquely determined overall.

Overall then, equations (i) and (ii) uniquely determine the family of morphisms $\{h_{(k)} : f_{(k)} \to Vg\}_{k\in\mathbb{N}}$ in $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$. ■

**The least fixed-point** To motivate that we seek the least fixed-point of iterating the pointed endofunctor $(c, S_n)$ of Lemmas A.2.16 and A.2.17, consider the following argument. At each step of iteration, $S_n$ formally adjoins solutions to $n$-dimensional lifting problems against the previous step (Lemma A.2.18). Given that we know that maps out of the object $\partial_n \rightarrowtail 2_n$ into $P$ of a filtered colimit $P(\operatorname{colim}_k S^k_n f)$ factor through some inclusion

$$P(S^i_n f) \rightarrowtail P(\operatorname{colim}_k S^k_n f)$$

(Lemma A.1.5 (iv) and pullback and $T$ preserve filtered colimits), we see that the colimit should have canonical solutions because we only ever need to go "one step further" to solve any problem posed. Finally, this construction of the least fixed-point has a universal property as, by Lemma A.2.22, every step of applying $S_n$ uniquely extends maps out of the previous step which preserve the canonical solutions we have chosen.

This argument is essentially the construction of the desired left adjoint $F_n$, and the remainder of this section is devoted to making rigorous these claims.

Recall that by Lemma A.1.3 (i), each category $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ has filtered colimits and so in particular each category of endofunctors has filtered colimits computed point-wise. Thus we are justified in making the following definition.

**Definition A.2.24.** The pointed endofunctor $(\eta, T_n)$ on $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ is defined to be the endofunctor arising as the colimit of the following $\omega$-indexed diagram of endofunctors,

$$\mathrm{id}_{(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}} \overset{c}{\Longrightarrow} S_n \overset{c_{S_n}}{\Longrightarrow} S_n^2 \overset{c_{S_n^2}}{\Longrightarrow} S_n^3 \Longrightarrow \cdots , \tag{A.2.25}$$

whose pointing is given by the first leg of the cocone, $\eta_n :\equiv \iota_0 \colon \mathrm{id} \Rightarrow T_n$. ◁

In the terminology of [Lei09, App. D], this is a nested colimit and the endofunctor thereby obtained bears the classical hallmarks of a least fixed-point.

**Lemma A.2.26.** *Each coprojection $S_n^k \Rightarrow T_n$ is monic, in particular the pointing $\eta_n$ of the endofunctor $T_n$ is monic. In addition, both $T_n$ and $\eta_n$ are the identity in all dimensions but $n$.*

*Proof.* Recall that the colimit for $T_n$ is computed point-wise, and filtered colimits in $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$ are created by $V_n$ in $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ (Lemma A.1.3 (i)). By the proof of Lemma 5.2.6, the category $(T\text{-Grph}/C)^{\mathrm{PRF}}_{-1} = T\text{-Grph}/C$ is a presheaf topos, and here coprojections for nested sequences are monic. Because each $V_n$ is faithful, monomorphisms are reflected from $T$-Grph/$C$.

Let us take this argument further still. Filtered colimits in a presheaf topos are themselves computed point-wise, and so by examining the explicit formulation Lemma A.1.4 (i) of filtered colimits in Set, as well as the details of our construction of $S_n$, we see that both $T_n$ and $\eta_n$ are the identity in all dimensions but $n$. ■

We may now translate our intuitions above into the following claims about $T_n$.

**Lemma A.2.27.** *The endofunctor $T_n$ on $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ factors through the forgetful functor $V_n \colon (T\text{-Grph}/C)^{\mathrm{PRF}}_n \to (T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$ as $T_n = V_n F_n$ where $F_n$ is trivial in all dimensions but $n$.*

*Proof.* For $f \in \mathrm{ob}(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$, we shall prove that $T_n f$ may in fact be equipped with the missing data of solutions to $n$-dimensional lifting problems against $PT_n f$, and so is an object of $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$ which we shall call $F_n f$. As $T_n$ acts trivially in all dimensions but $n$ (Lemma A.2.26), so $F_n$ inherits the same property. Then, for each morphism $h \colon f \to g$ of $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$, we will show that the morphism $T_n h$ is actually a morphism $F_n f \to F_n g$ in the category $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$. As compositions and identities of $(T\text{-Grph}/C)^{\mathrm{PRF}}_n$ are precisely those of the category $(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}$, in this way we shall have produced the claimed factorisation of functors $T_n = V_n F_n$.

To endow $T_n f$ with the data of solutions to $n$-dimensional lifting problems we intend to use Lemma A.1.7 as applied to the filtered colimit diagram (A.2.25) which defines $T_n$. Let us therefore verify each hypothesis of that lemma.

By Lemma A.1.5 (i) we know that any lifting problem $(\partial\varphi, \Phi)$ against $PT_n f$ factors through some index $k \in \mathrm{ob}\,\omega$ via the inclusion $PS_n^k f \rightarrowtail T_n f$. Whenever this happens, by Lemma A.2.18 we know that there is a canonical solution $\ulcorner \partial\varphi_k, \Phi_k \urcorner^k$ to the lifting problem $(Pc_{(k)})(\partial\varphi_k, \Phi_k)$ against $PS_n^{k+1} f$, whence the first hypothesis holds. For the second, observe that any non-trivial morphism $(k+1) \to (k+2+m)$ of $\omega$ factors uniquely as $(k+1) \to (k+2) \to \dots \to (k+2+m)$ so that it is enough to check the condition for only $(k+1) \to (k+2)$. In this case, the second hypothesis reduces precisely to the statement of Lemma A.2.20. Thus Lemma A.1.7 assures us that there is a well defined function $\kappa_{T_n f}$ solving all $n$-dimensional lifting problems against $PT_n f$. We therefore recast $T_n f$ as the object $F_n f \in (T\text{-Grph}/C)_n^{\mathrm{PRF}}$ equipped as it is with this data, for which it is clear that $T_n f = V_n F_n f$.

To conclude the proof it remains to establish that $T_n h$ is a morphism $F_n f \to F_n g$ of $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$, which in turn entails checking only that $PT_n h$ is a morphism of solutions to $n$-dimensional lifting problems. Given that $\kappa_{T_n f}$ is defined to be the inclusion of the solution $\ulcorner \partial\varphi_k, \Phi_k \urcorner^{f(k)}$, the condition that $T_n h$ preserve this solution may be seen to amount to the assertion of Lemma A.2.21. Thus $T_n = V_n F_n$ is a factorisation as claimed. ∎

**Lemma A.2.28.** *The pointing $\eta_n \colon \mathrm{id}_{(T\text{-Grph}/C)_{n-1}^{\mathrm{PRF}}} \Rightarrow V_n F_n$ is the unit of an adjunction $F_n \dashv V_n$, and is the identity in all dimensions but n. In addition, the counit of the adjunction is also the identity in all dimensions but n.*

*Proof.* Let us demonstrate the appropriate universal property, to which end we fix a morphism $h \colon f \to V_n g$ of $(T\text{-Grph}/C)_{n-1}^{\mathrm{PRF}}$. Lemma A.2.22 allows us to extend this to a cocone $\{h_{(k)} \colon S^k f_n \to V_n g\}_{k \in \mathbb{N}}$ under the diagram (A.2.25). Thus there exists a unique morphism $\overline{h} \colon T_n f \to V_n g$ in the category $(T\text{-Grph}/C)_{n-1}^{\mathrm{PRF}}$ factoring cocones appropriately. We shall prove that this morphism $\overline{h}$ is in fact a morphism $F_n f \to g$ of $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$, and moreover is the unique such satisfying $h = \overline{h}\eta_{n,f}$.

By our proof of Lemma A.2.27, we know that solutions to lifting problems $(\partial\varphi, \Phi)$ against $PF_n f$ are given by the inclusions $(P\iota_{(k)})^1 \ulcorner \partial\phi_k, \Phi_k \urcorner^k$ to the factored problems $(\partial\varphi, \Phi) = \iota_{(k)}(\partial\varphi_k, \Phi_k)$. Lemma A.2.22 therefore assures us of the third equality in the following calculation,

$$\begin{aligned}
(P\overline{h})^1 \kappa_{F_n f}(\partial\varphi, \Phi) &= (P\overline{h})^1 \left( (P\iota_{(k)})^1 \ulcorner \partial\phi_k, \Phi_k \urcorner^k \right) = (Ph_{(k)})^1 \ulcorner \partial\phi_k, \Phi_k \urcorner^k \\
&= \kappa_g \left( (Ph_{(k)})(\partial\varphi_k, \Phi_k) \right) = \kappa_g \left( (P\overline{h})(P\iota_{(k)})(\partial\varphi_k, \Phi_k) \right) \\
&= \kappa_g \left( (P\overline{h})(\partial\varphi, \Phi) \right) \ ,
\end{aligned}$$

which demonstrates that $P\overline{h}$ is indeed a morphism of $\mathrm{SCon}_{\leq n}\widehat{\mathbb{G}}$, so that we have $\overline{h} \colon F_n f \to g$ in $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$. Certainly $h = \overline{h}\eta_{n,f}$ by definition, so let us turn our attention to uniqueness.

Suppose given another morphism $\widehat{h} \colon F_n f \to g$ of $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$ which obeys $h = \widehat{h}\eta_{n,f}$. By the same argument as above we see that it must be the case that the cocone induced by pre-composition, $\{\widehat{h}_{(k)} :\equiv \widehat{h}\iota_{(k)}\}_{k \in \mathbb{N}}$, is a family satisfying both conditions in Lemma A.2.22 so that in fact $\widehat{h}_{(k)} = h_{(k)}$ for all $k$ and thus $\widehat{h} = \overline{h}$.

The first part of the last claim is Lemmas A.2.26 and A.2.27. By definition the counit $\varepsilon_n$ is induced by the universal property of $F_n V_n g$ applied to the cocone which is the extension (Lemma A.2.22) of the identity morphism $\mathrm{id} \colon V_n g \to V_n g$

in $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$. By unravelling definitions and applying the argumentation of Lemma A.2.26 we see that $\varepsilon_n$ is the identity in all dimensions but $n$. ■

## A.3. The proof

At last we may give a proof of the theorem.

**Theorem 5.2.3.** *The functor* $V\colon (T\text{-Grph}/C)^{\mathrm{PRF}} \to T\text{-Grph}/C$ *is monadic and finitary for every* $T$*-graph* $C$.

We will follow precisely the sketch we gave in the introduction to this appendix.

*Proof.* Recall Definition A.1.2, and consider the below $\omega^{\mathrm{op}}$ diagram of functors.

$$\cdots \xrightarrow{V_2} (T\text{-Grph}/C)_1^{\mathrm{PRF}} \xrightarrow{V_1} (T\text{-Grph}/C)_0^{\mathrm{PRF}} \xrightarrow{V_0} (T\text{-Grph}/C)_{-1}^{\mathrm{PRF}} = T\text{-Grph}/C$$

Inspection reveals that a limiting object for this diagram is $(T\text{-Grph}/C)^{\mathrm{PRF}}$, and in particular the projection to $(T\text{-Grph}/C)_{-1}^{\mathrm{PRF}}$ is the functor $V$ at issue.

By Lemmas A.1.3 and A.2.28 each functor $V_n$ in the $\omega^{\mathrm{op}}$ diagram above is finitary and monadic. This has two important consequences. First, as each morphism is an isofibration and $\omega^{\mathrm{op}}$ is a suitable indexing shape we may conclude that $(T\text{-Grph}/C)^{\mathrm{PRF}}$ is in fact a pseudo-limit of the same diagram. In particular therefore, limits in $(T\text{-Grph}/C)^{\mathrm{PRF}}$ are jointly created by the diagram, and $V$ is continuous.

Next, because $T$-Grph/$C$ is locally finitely presentable by Lemma 5.2.6, inductively we see that each $(T\text{-Grph}/C)_n^{\mathrm{PRF}}$ is locally finitely presentable for it is equivalent to the category of algebras for a finitary monad [AR94, Rem. 2.75]. Ergo the pseudo-limit $(T\text{-Grph}/C)^{\mathrm{PRF}}$ is locally finitely presentable [Bir84, Thm. 2.17].

Finally, by Lemma A.1.3 (i) we know that $V$ creates filtered colimits and is finitary. Thus $V$ is a continuous and finitary functor between locally finitely presentable categories and so has a left adjoint $F$ by the adjoint functor theorem [Bor94b, Thm. 5.5.7]. To conclude, Lemma A.1.3 (ii) shows that the functor $V$ satisfies the monadicity conditions. ■

## A.4. Notes

**An explicit description of the left adjoint** It may be disappointing to find that, although we have given an explicit construction of each left adjoint $F_n \dashv V_n$, we have not described explicitly the overall left adjoint $F$ and instead relied on the abstract machinery of locally presentable categories. It may be possible to adapt the work of [Che03a] to give just such a direct construction, by leveraging aspects of our left adjoints peculiar to the situation at hand. For instance, we suggest the following approach.

**Conjecture A.4.1** (Adjunction diagonalisation). *Suppose given a string of adjunctions*

$$\begin{tikzcd} \cdots \quad (T\text{-Grph}/C)^{\mathrm{PRF}}_1 \arrow[r, bend right, "V_1"'] \arrow[r, phantom, "\perp"] & (T\text{-Grph}/C)^{\mathrm{PRF}}_0 \arrow[l, bend right, "F_1"'] \arrow[r, bend right, "V_0"'] \arrow[r, phantom, "\perp"] & (T\text{-Grph}/C)^{\mathrm{PRF}}_{-1} \arrow[l, bend right, "F_0"'] \end{tikzcd},$$

*such that each functor $F_n$ is trivial in all ways on cells in dimensions $k \neq n$, and the same is true of the unit and co-unit. Let us write $F_{\leq n}$ for the following composite.*

$$F_{\leq n} :\equiv (T\text{-Grph}/C)^{\mathrm{PRF}}_n \xleftarrow{F_n} (T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1} \xleftarrow{F_{n-1}} \cdots \xleftarrow{F_0} (T\text{-Grph}/C)^{\mathrm{PRF}}_{-1}$$

*Then*

*(i) the assignment of $T$-graphs $f \colon A \to C$ to the following families of sets and functions indexed by dimensions $n \in \mathbb{N}$*

$$\begin{tikzcd} (F_{\leq n}A^0)_n \arrow[d, "(F_{\leq n}f^0)_n"'] & (F_{\leq n}A^1)_n \arrow[l, "(F_{\leq n}A^\ell)_n"'] \arrow[r, "(F_{\leq n}A^r)_n"] \arrow[d, "(F_{\leq n}f^1)_n"] & (TF_{\leq n}A^0)_n \arrow[d, "(TF_{\leq n}f^0)_n"] \\ C^0 & C^1 \arrow[l, "C^\ell"] \arrow[r, "C^r"'] & TC^0 \end{tikzcd},$$

*determines a functor $F \colon T\text{-Grph}/C \to (T\text{-Grph}/C)^{\mathrm{PRF}}$, and*

*(ii) the functor $F$ participates in an adjunction*

$$\begin{tikzcd} (T\text{-Grph}/C)^{\mathrm{PRF}} \arrow[r, bend right, "V"'] \arrow[r, phantom, "\perp"] & T\text{-Grph}/C \arrow[l, bend right, "F"'] \end{tikzcd}.$$

**Why this construction of each left adjoint?** The curious reader may wonder whether a different route to $F_n$ might have been charted. Such a reader might point out that the inclusions $\iota^{1,0}_A$ of Definition A.2.5 are also natural for $S_n$, and so we might have considered some colimit of $S_n$ over the pointing $\iota_A$ instead of $c$. However, this family of inclusions is not coherent with respect to preservation of lifting problems.

Observe that under $\iota^1_{(k)}$ each $A^1_{(k)}$ is embedded into its copy in $A^1_{(k+1)}$. Thus when we consider $\iota^1_{(k+1)} \colon A^1_{(k+1)} \rightarrowtail A^1_{(k+2)}$ we see that the lifting problems against $Pf_{(k)}$, which are $n$-cells of $L^1(f_{(k)}) \rightarrowtail A^1_{(k+1)}$, do not map to their appropriate copies in $L^1(f_{(k+1)}) \rightarrowtail A^1_{(k+2)}$, but instead are considered as cells $A^1_{(k+1)} \rightarrowtail A^1_{(k+2)}$. In this way, under a colimiting process for $(\iota, S_n)$, the least fixed-point would have countably many *distinct* copies of each lifting problem faced at any stage. To circumvent this issue and to give the correct fixed-point, we therefore made use of coherent inclusions.

This is not to say that there is no way to use $(\iota, S_n)$. Should we identify all of these extra copies of lifting problems at each stage, then an appropriate colimit might be reached. It is possible that a colimit of a diagram such as

$$\begin{tikzcd}
\mathrm{id}_{(T\text{-Grph}/C)^{\mathrm{PRF}}_{n-1}} \arrow[r, Rightarrow, "\iota"] & S_n \arrow[r, Rightarrow, shift left, "\iota_{S_n}"] \arrow[r, Rightarrow, shift right, "S_n\iota"'] & S_n^2 \arrow[r, Rightarrow, shift left=2, "\iota_{S_n^2}"] \arrow[r, Rightarrow] \arrow[r, Rightarrow, shift right=2, "S_n^2\iota"'] & S^3 \arrow[r, Rightarrow, shift left=3, "\iota_{S_n^3}"] \arrow[r, phantom, "\vdots"] \arrow[r, Rightarrow, shift right=3, "S_n^3\iota"'] & \cdots
\end{tikzcd},$$

perhaps a more readily familiar form of transfinite constructions of free monads, would yield the correct result. However, this colimit is not filtered and so proving its existence would constitute a non-trivial departure from the necessary work done in Lemma A.1.3 (i).

**Comparisons to other work** A reasonable criterion of the correctness of our adjunction might be a comparison to the work of [Che03a], itself an expansion for the results of [Lei09, App. G]. Unfortunately, on the surface of things, there does not appear to be an obvious way to compare these results. There is an evident functor Coll $:\equiv \widehat{\mathbb{G}}/T1 \to T\text{-Grph}/1$ which appends the missing leg to 1, but it is not a left adjoint—as seen by comparing initial objects. Worse still, there appears to be no good way to directly compare the codomain of the left adjoint Coll $\to$ CC there given, the category whose objects are collections equipped with contractions, to our category $(T\text{-Grph}/1)^{\mathrm{PRF}}$. Proof-relevant fibrations over 1 have the extra data of solutions to zero-dimensional lifting problems, and without constructing yet further functors the only easy comparison is given by a span CC $\to \mathscr{A} \leftarrow (T\text{-Grph}/1)^{\mathrm{PRF}}$ where $\mathscr{A}$ is like $(T\text{-Grph}/1)^{\mathrm{PRF}}$ but without the requirement for zero-dimensional lifting problem solutions. Of course this is not satisfactory, and does not yield anything enlightening. It is quite possible that there is some suitable functor CC $\to (T\text{-Grph}/1)^{\mathrm{PRF}}$ that yields a more interesting comparison between our left adjoint and the one Coll $\to$ CC, but we leave this possibility unaddressed for the time-being.

# B. Internalisation, enrichment, and $T$-categories

This appendix attempts to substantiate the apparatus of Section 5.4 by situating it as an example of a broader theory of enrichment vs. internalisation and a comparison of the effects of such processes on algebras. To that end we shed light on a few coincidences of theory between these, but our treatment is far from exhaustive. Doubtlessly there are even greater levels of generality which encompass our observations, and we attempt to suggest such avenues at the end in Appendix B.4

We believe that the material presented here, stated as it is at the level of virtual double categories and $T$-categories and with an emphasis on algebras, is new on the whole but special cases of some results recover known correspondences. For instance, Example B.1.8 specialises our work to the case considered in [CFP17] and explored in a higher-dimensional context in [FG22]. While we elaborate on our inspiration at the end of this chapter, we are grateful for a direct proof of a special case result—indeed a motivating example—provided to us by Lyne Moser.

In the coming sections we will freely make use of various aspects of the theory of virtual double categories, originally developed under the name fc-multicategory in [Lei99]. The content of that document is subsumed by [Lei09, §5], and surveyed and extended in [CS09] where fc-multicategories are treated under the present name. We will aim to recall the salient aspects of the theory as we use them, but the curious reader is directed to these references, and especially the notationally matched work [CS09].

For context in the below, let us fix a category $\mathscr{C}$ which is infinitary extensive, cartesian, and has chosen pullbacks. On this category let us fix a cartesian, coproduct-preserving monad $(T,\mu,\eta)$. While not all results on depend on all of these adjectives, we find it easier to assume this throughout.

With this in mind, the structure of this appendix is as follows. In Appendix B.1 we recall the notion of enrichement in a virtual double category and in particular enrichment in the horizontal Kleisli double category of $T$, $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$. We show how $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$-categories include familiar examples, and then prove that there is a generalised internalisation functor $\mathrm{Int}\colon \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)\text{-Cat} \to T\text{-Cat}$ sending enriched categories to internal categories. In addition we observe that there is not one, but a family of opposed externalisation functors and offer a conjectured adjunction between internalisation and externalisation. Then in Appendix B.2 we give a definition of algebra of an $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$-category $\mathcal{C}$ and focus our attention on how the algebras of $\mathcal{C}$ compare to algebras in the internal sense for $\mathrm{Int}\,\mathcal{C}$. We find in Lemma B.2.7 that internal algebras for the internalisation are a reflective subcategory of the external algebras. Then in Appendix B.3 we add another point of comparison: we observe that there is a functor from $\mathscr{C}^T$-categories to $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$-categories and moreover describe a faithful adjunction in Lemma B.3.11 for comparing algebras under the action of this functor. To conclude the work we prove Theorem B.3.15, a very general form of the Grothendieck construction which specialises precisely to the classical statement in simple circumstances: algebras for an enriched category are equivalent to internal discrete opfibrations. Finally in Appendix B.4 we offer some suggestions for avenues of further exploration, and speculate on ways to improve and unify the results presented.

Although as we mention we are motivated by the application of Theorem B.3.15 to a particular case, we hope that the following exploration of the confluence of the theories of enrichment, $T$-categories, and algebras will prove of independent interest to the reader.

## B.1. Internalisation for virtual double categories of spans

Following [Lei99, §2][12] let us recall the notion of a category enriched in a virtual double category. In the below definition we have opted to elaborate the abstract version given in §2.2 *ibid.*

**Definition B.1.1.** Given a set $S$, the **vertically discrete**, **indiscrete virtual double category** $\mathrm{Ind}_{\|} S$ is the virtual double category whose set of objects is $S$, whose only vertical morphisms are trivial, and whose horizontal morphisms and cells are indiscrete. That is, there is a unique horizontal morphism between every pair of objects, and a unique cell in every boundary.

Given a virtual double category $\mathbb{V}$, **the category** $\mathbb{V}$-Cat **of** $\mathbb{V}$**-enriched categories**, $\mathbb{V}$**-functors**, **and** $\mathbb{V}$**-natural transformations** is defined to be the lax-type comma category $\mathbb{V}\text{-Cat} :\equiv (\mathrm{Ind}_{\|} \downarrow_{\ell} \mathbb{V})$ of the span

$$\mathrm{Set} \xrightarrow{\mathrm{Ind}_{\|}} \mathrm{vDbl} \xleftarrow{\mathbb{V}} 1$$

where Set is considered as a discrete 2-category. In detail, a $\mathbb{V}$-category $\mathcal{C}$ is a functor $\mathcal{C} \colon \mathrm{Ind}_{\|}(\mathrm{ob}\,\mathcal{C}) \to \mathbb{V}$. A $\mathbb{V}$-functor $F \colon \mathcal{C} \to \mathcal{D}$ is a function $\mathrm{ob}\,F \colon \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ together with a transformation $F \colon \mathcal{C} \Rightarrow \mathcal{D} \circ \mathrm{Ind}_{\|}(\mathrm{ob}\,F)$, and composition of $\mathbb{V}$-functors is as dictated by pasting of such transformations. ◁

In more elementary terms, a $\mathbb{V}$-category $\mathcal{C}$ comprises a set $\mathrm{ob}\,\mathcal{C}$ of objects, as well as for each object $c \in \mathrm{ob}\,\mathcal{C}$ an object $\mathcal{C}(c)$ in $\mathbb{V}$, and for each pair of objects a "hom" $\mathcal{C}(c,d) \colon \mathcal{C}(d) \nrightarrow \mathcal{C}(c)$, and so on. This definition is a generalisation of the definition of enrichment in a monoidal category, enrichment in a bicategory, and enrichment in a multicategory. In fact each of these notions may be recovered from enrichment in virtual double category when by appropriately degenerate the data of $\mathbb{V}$. For instance, if $\mathbb{V}$ has only a single object and a single vertical morphism—so that it is essentially a multicategory—enrichment in $\mathbb{V}$ coincides with the notion of enrichment in a multicategory, which itself coincides with enrichment in a monoidal category when the multicategory has tensors.

[12] recounted also in [Lei02, §2]

For the purposes of this section, we are interested in enriching in a virtual double category of $T$-spans.

**Definition B.1.2.** The **horizontal Kleisli virtual double category of** $T$, written $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$ in the naming convention of [CS09], is the virtual double category whose objects and horizontal morphisms are those of $\mathrm{Span}(\mathscr{C}, T)$ (Definition 2.2.2), whose vertical morphisms are those of $\mathscr{C}$, and whose cells are given as in [CS09, Def. 4.1]. ◁

Therefore we may readily consider categories enriched in $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$, and for these categories we now demonstrate a pleasant connection to the theory of $T$-categories. To render readable our claims, let us introduce some notation for this special case.

*Notation* B.1.3. Given a $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category $\mathcal{C}$, let us write

- $\mathcal{C}(a)$ for the set which is the image of the object $a \in \mathrm{ob}\,\mathcal{C}$,
- $f^{\ell}_{a,b}$ and $f^{r}_{a,b}$ for the legs of the span

$$\mathcal{C}(b) \xleftarrow{f^{\ell}_{a,b}} \mathcal{C}(a,b) \xrightarrow{f^{r}_{a,b}} T\,\mathcal{C}(a)$$

  which is the image of the unique horizontal arrow $a \nrightarrow b$ in $\mathrm{Ind}_{\shortparallel}\,\mathcal{C}$,
- $\circ_{a,b,c}$ for the below-right image of the below-left cell of $\mathrm{Ind}_{\shortparallel}\,\mathcal{C}$,

$$\begin{tikzcd} a \arrow[r, "\shortmid" marking] \arrow[d, equal] & b \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\exists!"] & c \arrow[d, equal] \\ a \arrow[rr, "\shortmid" marking] & & c \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(c) \arrow[r, "\mathcal{C}(b,c)", "\shortmid" marking] \arrow[d, equal] & \mathcal{C}(b) \arrow[r, "\mathcal{C}(a,b)", "\shortmid" marking] \arrow[d, phantom, "\circ_{a,b,c}"] & \mathcal{C}(a) \arrow[d, equal] \\ \mathcal{C}(c) \arrow[rr, "\mathcal{C}(a,c)"', "\shortmid" marking] & & \mathcal{C}(a) \end{tikzcd}$$

- $\mathsf{refl}_a \colon \mathcal{C}(a) \to \mathcal{C}(a,a)$ for the image of the unique nullary cell from $a$ to $a \nrightarrow a$.

◁

*Remark* B.1.4. Note that this form of enrichment is somehow "unbiased", in that we supply for each list of $n$-objects a matching composition operation on their images. The fact that $\mathrm{Ind}_{\shortparallel}(\mathrm{ob}\,\mathcal{C})$ is indiscrete horizontally and on cells means that all possible coherences between these composition operations hold simultaneously. In particular we know that associative binary composites which are unital with respect to nullary composites present all $n$-ary composities, and in this sense [Lei02, Def. 2] is a presentation of this definition. ◁

**Example B.1.5**

Categories enriched in $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$ include many familiar examples. When $T = \mathrm{id}$, enriched categories $\mathcal{C}$ for which $\mathcal{C}(a) = 1$ for all $a \in \mathrm{ob}\,\mathcal{C}$ are categories enriched in $(\mathscr{C}, \times, 1)$. A classical example of this example is $\mathscr{C} = \mathrm{Cat}$, such enriched categories are precisely the 2-categories. When $\mathscr{C} = \widehat{\mathbb{G}}$ and $T$ is the free strict $\omega$-category, enriched categories for which $\mathcal{C}(a) = 1$ are the Coll-categories: a one-object Coll-category is a globular operad, as defined in [Lei09, §8]. See also [Bre18] for an enriched perspective.

This is part of a larger class of examples which make use of the following observation: a one-object $\mathbb{V}$-enriched category is a monoid in $\mathbb{V}$ [CS09, Def. 2.8]. When $T$ is the free category monad on graphs, a one-object $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category is a virtual double category. When $T$ is the free monoid monad on $\mathbb{S}\mathrm{pan}\,\mathrm{Set} \cong \mathrm{Set}\text{-}\mathbb{M}\mathrm{at}$, a one-object $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category is an ordinary multicategory. For other examples see, for instance, [CS09, §4].

With this notation we are ready to construct a $T$-category from every category enriched in $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$. So far we have not made use of our assumptions about coproducts and their interactions with pullbacks and $T$, but they now become crucial.

**Definition B.1.6.** Let $(\mathrm{ob}\,\mathcal{C}, \mathcal{C}(-,-), \mathsf{comp}, \mathsf{refl})$ be a $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category, the **underlying $T$-graph of the internalisation of** $\mathcal{C}$, written $\mathrm{Int}\,\mathcal{C}$, is the $T$-graph of defined by the following diagram.

$$\begin{tikzcd}
\coprod_{b \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(b) & \coprod_{a,b \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(a,b) \arrow[l, dashed, "c"'] \arrow[r, dashed, "d"] & T\left(\coprod_{a \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(a)\right) \\
\mathcal{C}(b) \arrow[u, "\iota_b"] & \mathcal{C}(a,b) \arrow[u, "\iota_{a,b}"] \arrow[l, "f^{\ell}_{a,b}"] \arrow[r, "f^{r}_{a,b}"'] & T\mathcal{C}(a) \arrow[u, "T\iota_a"']
\end{tikzcd}$$

Through the evident action of $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-functors $F\colon \mathcal{C} \to \mathcal{D}$ this assignment extends to a functor $\mathrm{Int}\colon \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-Cat} \to T\text{-Grph}$. ◁

**Lemma B.1.7.** *Let* $(\mathrm{ob}\,\mathcal{C}, \mathcal{C}, \mathsf{refl}, \mathsf{comp})$ *be an* $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$*-category, then the underlying $T$-graph of the internalisation of* $\mathcal{C}$ *is equipped with a $T$-category structure. Moreover, this assignment of objects* $\mathcal{C} \mapsto \mathrm{Int}\,\mathcal{C}$ *extends to a functor*

$$\mathrm{Int}\colon \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-Cat} \to T\text{-Cat}\,.$$

*Proof.* Let us address the first claim. We will begin by giving the structural morphisms of $\mathsf{refl}\colon \mathrm{id}_{(\mathrm{Int}\,\mathcal{C})_0} \Rightarrow \mathrm{Int}\,\mathcal{C}$ and then $\mathsf{comp}\colon \mathrm{Int}\,\mathcal{C} \circ \mathrm{Int}\,\mathcal{C} \Rightarrow \mathrm{Int}\,\mathcal{C}$, before proving that they satisfy the necessary relations.

In the first case we see that a satisfactory definition of $\mathsf{refl}$ as below-left is given by the below-right diagram.

$$
\begin{tikzcd}
\coprod_{b\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(b) & \coprod_{a,b\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(a,b) \arrow[l, "c"'] \arrow[d, "d"] \\
\coprod_{c\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(c) \arrow[u, equal] \arrow[ur, "\mathsf{refl}"] \arrow[r, "\eta"'] & T\left(\coprod_{a\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(a)\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_{a\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(a) \arrow[r, dashed, "\mathsf{refl}"] & \coprod_{a,b\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(a,b) \\
\mathcal{C}(a) \arrow[u, "\iota_a"] \arrow[r, "\mathsf{refl}_a"'] & \mathcal{C}(a,a) \arrow[u, "\iota_{a,a}"']
\end{tikzcd}
$$

To give comp we must first compute $\mathrm{Int}\,\mathcal{C} \circ \mathrm{Int}\,\mathcal{C}$. By using the fact that $\mathscr{C}$ is an infinitary extensive category, that $T$ preserves coproducts, and some pullback lemmas, we may realise this composite as the below-indicated pullback square.

$$
\begin{tikzcd}
\coprod\limits_{a,b,c\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(b,c) \circ_b \mathcal{C}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T\Big(\coprod\limits_{b,c\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(a,b)\Big) \arrow[d, "Tc"] \\
\coprod\limits_{a,b\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(b,c) \arrow[r, "d"'] & T\Big(\coprod\limits_{b\in\mathrm{ob}\,\mathcal{C}} \mathcal{C}(b)\Big)
\end{tikzcd}
$$

From here there is an evident morphism $\mathsf{comp}\colon \mathrm{Int}\,\mathcal{C} \circ \mathrm{Int}\,\mathcal{C} \to \mathrm{Int}\,\mathcal{C}$ given by the maps $\mathsf{comp}_{a,b,c}\colon \mathcal{C}(b,c) \circ_b \mathcal{C}(a,b) \to \mathcal{C}(a,c)$ on each coproduct component. A lengthy, if mechanical computation verifies that this comp morphism is associative and unital with respect to refl given above, essentially because $\mathcal{C}$ is a $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$-category. Functoriality follows through similarly elaborate, if straightforward calculations. ■

**Example B.1.8**

When $T = \mathrm{id}$ is the identity monad we obtain some substantial simplifications and the result is applicable to the study of internal category theory. As we point out in Example B.1.5, categories enriched in $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$ in the sense of Definition B.1.1 above include in particular categories enriched in the monoidal category $(\mathscr{C},\times,1)$ in the ordinary sense—those for which $\mathcal{C}(a) = 1$ for all $a$. For these ordinary $\mathscr{C}$-categories, our functor Int coincides with the internalisation of $\mathscr{C}$-categories into $\mathrm{Cat}(\mathscr{C}) = T\text{-}\mathrm{Cat} = \mathcal{V}\,\mathbb{K}\mathrm{Mod}(\mathscr{C},\mathrm{id})$ studied in [CFP17, §4]. A familiar example of this example is obtained when $\mathscr{C} = \mathrm{Cat}$, viz., the axial embedding of 2-categories into double categories.

*Remark* B.1.9. As noted in [CFP17, §1], it is not generally the case that the "ordinary" internalisation functor $\mathrm{Int}\colon (\mathscr{C},\times,1)\text{-}\mathrm{Cat} \to \mathrm{Cat}(\mathscr{C})$ is faithful. As this functor is a special case of our Int, we know not to expect faithfulness of our construction. ◁

From this example we might expect that internalisation has a right adjoint, the "externalised enriched category" construction exhibited in [CFP17, §4.2]. In our context then, this should amount to a functor $\mathcal{V}\,\mathbb{K}\mathrm{Mod}(\mathscr{C},T) \to \mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)\text{-}\mathrm{Cat}$. However, unlike in the enriched-in-monoidal-category treatment of the matter, here we have more flexibility. Let us temporarily write $\mathrm{Ext}\,\mathbb{A}$ for the supposed value obtained by externalising a $T$-category $\mathbb{A}$. In the simplified case of the example above, we would have fixed $(\mathrm{Ext}\,\mathbb{A})(x) = 1$ for all objects $x \in \mathrm{ob}\,\mathrm{Ext}\,\mathbb{A}$—but this is an artificial constraint. Indeed, as we shall see, there is not one functor in the other direction, but an entire family. For related reasons, we must be more careful in seeking a left adjoint to internalisation. As we shall shortly see, we do not need to know whether Int is a right adjoint in order to reason about its effects on algebras. Nevertheless we cannot help but explore briefly the notion of externalisation.

**Definition B.1.10.** Let $\mathcal{C}$ be a $\mathbb{H}$-Kl($\mathbb{S}$pan $\mathscr{C}, T$)-category. The $\mathcal{C}$-**fibred externalisation functor** $\mathrm{Ext}_{\mathcal{C}}$ is a functor $\mathrm{Ext}_{\mathcal{C}}\colon T\text{-Cat} \to \mathbb{H}\text{-Kl}(\mathbb{S}\text{pan}\,\mathscr{C}, T)\text{-Cat}/\mathcal{C}$ which will we shortly determine. On a $T$-category $\mathbb{A}$ with data $\mathbb{A} = (A^{0,1}, c, d, \mathsf{refl}, \mathsf{comp})$, the $\mathbb{H}$-Kl($\mathbb{S}$pan $\mathscr{C}, T$)-category $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ comprises the following data.

(i) The set $\mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ is the set $\coprod_{c\in\mathrm{ob}\,\mathcal{C}} \mathscr{C}(\mathcal{C}(c), A^0)$.

(ii) The value of the virtual double functor $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ at an element $(c, g_c\colon \mathcal{C}(c) \to A^0)$ is the object $\mathcal{C}(c)$.

(iii) Given two elements of $\mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$, $(a, g_a\colon \mathcal{C}(a) \to A^0)$ and $(b, g_b\colon \mathcal{C}(b) \to A^0)$, the value of $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ at the unique horizontal morphism $(a, g_a) \nrightarrow (b, g_b)$ is the $T$-span $g_{a,b}$ from $\mathcal{C}(b)$ to $\mathcal{C}(a)$ whose data is given as indicated in the following pullback in $\mathscr{C}$.

$$\begin{tikzcd}
\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}\big((a,g_a),(b,g_b)\big) \arrow[rrr, "\pi_A^{a,b}"] \arrow[d] \arrow[drr, dashed, "\langle g^{\ell}_{a,b}, g^{r}_{a,b}\rangle"] \arrow[dr, phantom, very near start, "\lrcorner"] & & & A^1 \arrow[d, "\langle c,d\rangle"] \\
\mathcal{C}(a,b) \arrow[r, "\langle f^{\ell}_{a,b}, f^{r}_{a,b}\rangle"'] & \mathcal{C}(b)\times T\mathcal{C}(a) \arrow[rr, "g_b \times T g_a"'] & & A^0 \times TA^0
\end{tikzcd}$$

(iv) The value of $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ at the unique nullary with domain $(a, g_a)$ is given by the universal property of $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}\big((a,g),(a,g)\big)$, $\mathsf{refl}^{\mathcal{C}}$, and $\mathsf{refl}^{\mathbb{A}}$ as follows.

$$\begin{tikzcd}
\mathcal{C}(a) \arrow[r, "a"] \arrow[dr, dashed, "\mathsf{refl}_{(a,g)}"] \arrow[ddr, bend right, "\mathsf{refl}^{\mathcal{C}}_a"'] & A^0 \arrow[rrr, "\mathsf{refl}^{\mathbb{A}}"] & & & A^1 \arrow[d, equal] \\
& \mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}\big((a,g),(a,g)\big) \arrow[rrr, "\pi_A"] \arrow[d] \arrow[dr, "\langle g^{\ell}_{a,a}, g^{r}_{a,a}\rangle"] & & & A^1 \arrow[d, "\langle c,d\rangle"] \\
& \mathcal{C}(a,a) \arrow[r, "\langle f^{\ell}_{a,a}, f^{r}_{a,a}\rangle"'] & \mathcal{C}(a)\times T\mathcal{C}(a) \arrow[rr, "Tg_a \times g_a"'] & & A^0 \times TA^0
\end{tikzcd}$$

(v) Finally the data of $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ on the unique cell in $\mathrm{Ind}_{\parallel}(\mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A})$ whose top boundary is $(a_{n+1}, g_{n+1}) \nrightarrow (a_n, g_n) \nrightarrow \ldots \nrightarrow (a_0, g_0)$ and whose bottom boundary is $(a_{n+1}, g_{n+1}) \nrightarrow (a_0, g_0)$ is given in two steps. First by the universal property of pullbacks we obtain a morphism $\pi_{\langle (a_i,g_i)\rangle}$

$$\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}\big((a_n,g_n),(a_{n+1},g_{n+1})\big) \circ_n \ldots \circ_1 \mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}\big((a_0,g_0),(a_1,g_1)\big) \to (A^1)^{\circ(n+1)} .$$

Then we compose this with the unambiguous $n$-ary composition morphism $\mathsf{comp}^{\mathbb{A}}_n\colon (A^1)^{\circ(n+1)} \to A^1$ to obtain a map into $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}((a_0,g_0),(a_{n+1},g_{n+1}))$ by the universal property thereof, whose projection to $A^1$ is $\mathsf{comp}^{\mathbb{A}}_n\,\pi_{\langle (a_i,g_i)\rangle}$ and whose projection to $\mathcal{C}(a_0, a_{n+1})$ factors through $\mathsf{comp}^{\mathcal{C}}$ on the $a_i$.

There is an evident functor $\mathrm{Ext}_{\mathcal{C}}\,\mathbb{A} \to \mathcal{C}$ given by projection, and it may be verified that universal property arguments extend the assignment $\mathbb{A} \mapsto \mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}$ to a functor $\mathrm{Ext}_{\mathcal{C}} \colon T\text{-Cat} \to \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-Cat}/\mathcal{C}$. ◁

**Example B.1.11**

When $T = \mathrm{id}$ and $\mathcal{C} = 1$ is the terminal $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category our externalisation functor $\mathrm{Ext}_1 \colon \mathrm{Cat}(\mathscr{C}) \to \mathscr{C}\text{-Cat}$ simplifies to match that of [CFP17, §4.2].

There is always a "co-unit" for this not-quite-adjunction, as we shall now claim.

**Lemma B.1.12.** *Let us write* $\Pi$ *for the projection*

$$\Pi \colon \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-Cat}/\mathcal{C} \to \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-Cat}\ .$$

*For each* $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$*-category* $\mathcal{C}$*, there is a natural transformation*

$$\varepsilon \colon \mathrm{Int}\,\Pi\,\mathrm{Ext}_{\mathcal{C}} \Rightarrow \mathrm{id}_{T\text{-Cat}}\ .$$

*Proof* (Sketch). Given a $T$-category $\mathbb{A}$, the composite $\mathrm{Int}\,\Pi\,\mathrm{Ext}_{\mathcal{C}}$ at $\mathbb{A}$ comprises at least the following data.

$$\begin{tikzcd}
\coprod\limits_{\substack{b \in \mathrm{ob}\,\mathcal{C} \\ \mathscr{C}(\mathcal{C}(b), A^0)}} \mathcal{C}(b) & \coprod\limits_{\substack{a,b \in \mathrm{ob}\,\mathcal{C} \\ g_a \in \mathscr{C}(\mathcal{C}(a), A^0) \\ g_b \in \mathscr{C}(\mathcal{C}(b), A^0))}} \mathrm{Ext}_C\,\mathbb{A}\big((a,g_a),(b,g_b)\big) \arrow[l, dashed, "c"'] \arrow[r, dashed, "d"] & T \coprod\limits_{\substack{a \in \mathrm{ob}\,\mathcal{C} \\ \mathscr{C}(\mathcal{C}(a), A^0)}} \mathcal{C}(a) \\
\mathcal{C}(b) \arrow[u, "\iota_{(b,g_b)}"] & \mathrm{Ext}_{\mathcal{C}}\,\mathbb{A}\big((a,g_a),(b,g_b)\big) \arrow[l, "g^{\ell}_{a,b}"] \arrow[r, "g^{r}_{a,b}"'] \arrow[u, "\iota_{(a,g_a),(b,g_b)}"] & T\,\mathcal{C}(a) \arrow[u, "T\iota_{(a,g_a)}"']
\end{tikzcd}$$

Note that there are canonical maps

$$[g_b]_{(b,g_b)} \colon \coprod_{\substack{b \in \mathrm{ob}\,\mathcal{C} \\ \mathscr{C}(\mathcal{C}(b), A^0)}} \mathcal{C}(b) \to A^0$$

$$[\pi^{a,b}_A] \colon \coprod_{\substack{a,b \in \mathrm{ob}\,\mathcal{C} \\ g_a \in \mathscr{C}(\mathcal{C}(a), A^0) \\ g_b \in \mathscr{C}(\mathcal{C}(b), A^0)}} \mathrm{Ext}_C\,\mathbb{A}\big((a,g_a),(b,g_b)\big) \to A^1\ ,$$

which form at least a morphism of $T$-graphs $\varepsilon_{\mathbb{A}} \colon \mathrm{Int}\,\Pi\,\mathrm{Ext}_C\,\mathbb{A} \to \mathbb{A}$. We claim that this is in fact a morphism of $T$-categories, and moreover natural with respect to morphisms thereof. The details of this proof are exorbitant in their minutiae, so we conclude our sketch here. ■

Unfortunately the unit does not present itself between these categories. Fix a $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category $\mathcal{D}$ and observe that the round trip $\mathrm{Ext}_{\mathcal{C}}\,\mathrm{Int}\,\mathcal{D}$ leaves us with an enriched category whose object set is given by

$$\mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathrm{Int}\,\mathcal{D} = \coprod_{c \in \mathrm{ob}\,\mathcal{C}} \mathscr{C}\big(\mathcal{C}(c), \textstyle\coprod_{d \in \mathrm{ob}\,\mathcal{D}} \mathcal{D}(d)\big) \ .$$

Because we have no information about how $\mathcal{C}(c)$ and $\mathcal{D}(d)$ relate, there is is no obvious function $\mathrm{ob}\,\mathcal{D} \to \mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathrm{Int}\,\mathcal{D}$ and it is here that we see the relative strength of the simplified setting of [CFP17].

However we need not constrain ourselves so severely. Note that an arbitrary $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-functor $F\colon \mathcal{D} \to \mathcal{C}$ would not obviously provide sufficient information to give a function $\mathrm{ob}\,\mathcal{D} \to \mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathrm{Int}\,\mathcal{D}$. Rather what the mathematics appears to be suggesting is that we focus our attention on the full sub-category of $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C}$ whose objects $F\colon \mathcal{D} \to \mathcal{C}$ satisfy the property that for all $d \in \mathrm{ob}\,\mathcal{D}$ there is a coincidence $\mathcal{C}(Fd) = \mathcal{D}(d)$. For these objects there *is* a function $d \mapsto \iota_{Fd}(\iota_d)$ of objects $\mathrm{ob}\,\mathcal{D} \to \mathrm{ob}\,\mathrm{Ext}_{\mathcal{C}}\,\mathrm{Int}\,\mathcal{D}$, and certainly the codomain of $\mathrm{Ext}_{\mathcal{C}}$ factors through this sub-category. We therefore pause here to make the following suggestion, for which we regret to provide no proof or further justification in the present document.

**Conjecture B.1.13.** *Let us write* $(\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C})_{\mathrm{vico}}$ *for the full sub-category of* $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C}$ *whose objects* $F\colon \mathcal{D} \to \mathcal{C}$ *satisfy the property that for all* $d \in \mathrm{ob}\,\mathcal{D}$ *there is a coincidence* $\mathcal{C}(Fd) = \mathcal{D}(d)$*—those enriched functors which have vertical identity components on objects in this sense.*

*Observe that* $\mathrm{Ext}_{\mathcal{C}}$ *factors through the inclusion*

$$(\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C})_{\mathrm{vico}} \rightarrowtail \mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C} \ ,$$

*and that we may pre-compose* Int *by the projection*

$$\Pi\colon (\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C})_{\mathrm{vico}} \to \mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat} \ .$$

*This having been done, we suggest that there is an adjunction*

$$\begin{tikzcd} \mathcal{V}\,\mathbb{K}\mathrm{Mod}(\mathscr{C}, T) \arrow[r, bend right, "\mathrm{Ext}_{\mathcal{C}}"'] \arrow[r, phantom, "\perp"] & (\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)\text{-}\mathrm{Cat}/\mathcal{C})_{\mathrm{vico}} \arrow[l, bend right, "\mathrm{Int}\circ\Pi"'] \end{tikzcd}.$$

## B.2. Comparing algebras

In some ways the point of internalisation is to grant access to the specific devices of $T$-category theory that would otherwise be inapplicable to $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-categories. However, we have already seen that the internalisation functor does not generally find itself embroiled in the expected adjunction, and so the wary reader may wonder about the utility of predicating further arguments on internalised objects. It is this reader that we hope to convince by proving that internalisation does preserve an important aspect of both theories: algebras.

On the one hand we already know well what the algebras of a $T$-category are (Theorem 2.2.11), so it remains to pin down a notion of algebra for an enriched category. Let us trace a path to a definition as we would classically, by beginning with bimodules. Our definitions are standard inasmuch such matters appear in the literature, and we match for instance those of [Mye20, §3].

**Definition B.2.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{V}$-categories. An $(\mathcal{C}, \mathcal{D})$-**bimodule** $M$ comprises the data of

(i) a horizontal morphism $M(y,x)\colon \mathcal{C}(x) \nrightarrow \mathcal{D}(y)$ for each pair $(x,y) \in \mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{D}$,

(ii) cells of the form

$$\begin{tikzcd} \mathcal{C}(x') \arrow[r, "{\mathcal{C}(x,x')}", "|" marking] \arrow[d, equal] & \mathcal{C}(x) \arrow[r, "{M(y,x)}", "|" marking] \arrow[d, phantom, "\rho_{x,x'}"] & \mathcal{D}(y) \arrow[d, equal] \\ \mathcal{C}(x') \arrow[rr, "|" marking, "{M(y,x')}"'] & & \mathcal{D}(y) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(x) \arrow[r, "{M(y',x)}", "|" marking] \arrow[d, equal] & \mathcal{D}(y') \arrow[r, "{\mathcal{D}(y,y')}", "|" marking] \arrow[d, phantom, "\lambda_{y,y'}"] & \mathcal{D}(y) \arrow[d, equal] \\ \mathcal{C}(x) \arrow[rr, "|" marking, "{M(y,x)}"'] & & \mathcal{D}(y) \end{tikzcd}$$

for each pair of objects $x, x' \in \mathrm{ob}\,\mathcal{C}$ and each pair of objects $y, y' \in \mathrm{ob}\,\mathcal{D}$,

such that the cells $\rho$ and $\lambda$ are compatible with each other and the $n$-ary composition cells $\circ$ of $\mathcal{C}$ and $\mathcal{D}$. We say that $\mathcal{D}$ acts contravariantly and $\mathcal{C}$ acts covariantly.

Given two $(\mathcal{C}, \mathcal{D})$-bimodules $M$ and $N$, a **bimodule transformation** $\alpha\colon M \Rightarrow N$ comprises the data of cells

$$\begin{tikzcd} \mathcal{C}(x) \arrow[r, "{M(y,x)}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\alpha_{y,x}"] & \mathcal{D}(y) \arrow[d, equal] \\ \mathcal{C}(x) \arrow[r, "|" marking, "{N(y,x)}"'] & \mathcal{D}(y) \end{tikzcd}$$

which are compatible with those cells $\rho^M, \lambda^M$ and $\rho^N, \lambda^N$ of $M$ and $N$. Bimodule transformations compose, and there is an identity bimodule transformation on each bimodule. As such we form the **category** $\mathrm{BiMod}(\mathcal{C}, \mathcal{D})$ **of** $(\mathcal{C}, \mathcal{D})$-**bimodules and transformations**. ◁

This definition of bimodules is appropriately functorial, as we now record.

**Lemma B.2.2.** *Given $\mathbb{V}$-functors $F\colon \mathcal{C}' \to \mathcal{C}$ and $G\colon \mathcal{D}' \to \mathcal{D}$, by pre-composition we obtain a restriction functor of bimodules* $\mathrm{BiMod}(F, G)\colon \mathrm{BiMod}(\mathcal{C}, \mathcal{D}) \to \mathrm{BiMod}(\mathcal{C}', \mathcal{D}')$.

*Let us emphasise the enriching category* $\mathbb{V}$ *and write* $\mathrm{BiMod}_{\mathbb{V}}(\mathcal{C},\mathcal{D})$. *Any virtual double functor* $H\colon \mathbb{V} \to \mathbb{W}$ *induces by post-composition a functor* $H_*\colon \mathbb{V}\text{-Cat} \to \mathbb{W}\text{-Cat}$. *As bimodules and transformations are defined equationally,* $H$ *additionally induces a functor* $\mathrm{BiMod}_H\colon \mathrm{BiMod}_{\mathbb{V}}(\mathcal{C},\mathcal{D}) \to \mathrm{BiMod}_{\mathbb{W}}(H_*\mathcal{C},H_*\mathcal{D})$ *of categories.*

*In addition, any virtual double transformation* $\alpha\colon H \Rightarrow G$ *between parallel virtual double functors* $H,G\colon \mathbb{V} \rightrightarrows \mathbb{W}$ *induces by composition in* $\mathbb{W}$ *a natural transformation* $\mathrm{BiMod}_\alpha\colon \mathrm{BiMod}_H \Rightarrow \mathrm{BiMod}_G$. *Let us write* vDbl *for the* 2*-category of virtual double categories, virtual double functors and virtual double transformations. All in all, taking bimodules extends to a* 2*-functor* $\mathrm{BiMod}\colon \mathrm{vDbl} \to \mathrm{Cat}$. ■

To imitate the classical definition of an algebra for a category, we might be tempted to declare that algebras for a $\mathbb{V}$-category $\mathcal{C}$ are $(\mathcal{C},1)$-bimodules where $1$ is a terminal $\mathbb{V}$-category. Setting aside temporarily the question of the existence of a terminal $\mathbb{V}$-category, it turns out that this definition does not suitably capture the ordinary sense of an algebra for an enriched category. To make sense of this, recall that we may cast a monoidal category $\mathcal{M}$ as degenerate virtual double category: there is one object $*$, no nontrivial vertical morphisms, the objects of $\mathcal{M}$ form the horizontal morphisms, and the cells are morphisms in $\mathcal{M}$ from the (say) left-associated tensor of the domain to the codomain.

**Counter-example B.2.3**

Whenever $(\mathcal{M},\otimes,I)$ has a terminal object, the category $\mathcal{M}$-Cat has a terminal object $1$. The object set $\mathrm{ob}\,1 = \{*\}$ is a terminal set, and $1(*,*) = 1$ a terminal object of $\mathcal{M}$. The rest of the data for $1$ is uniquely determined, and it is straightforward to verify that $1$ is a terminal $\mathcal{M}$-category. Now let $\mathcal{C}$ be an $\mathcal{M}$-category and consider the following.

If $1$ is not the monoidal unit $I$ then algebras for $\mathcal{C}$ are not necessarily $(\mathcal{C},1)$-bimodules. Recall that an algebra for $\mathcal{C}$ is a family $\{A_x\}_{x\in\mathrm{ob}\,\mathcal{C}}$ with action maps $\mathcal{C}(x,x')\otimes A_x \to A_{x'}$ which are appropriately unital and coherent. Observe that an $(\mathcal{C},1)$-bimodule $M$ includes in particular the data of a family of maps $\{M(*,x)\otimes 1 \to M(*,x)\}_{x\in\mathrm{ob}\,\mathcal{C}}$. Our claim is that these maps may be genuinely more structure than an ordinary algebra would otherwise have.

In the cocartesian monoidal category $(\mathrm{Set},\coprod,\emptyset)$, $1$ is a terminal set and so giving a map $M(*,x)\coprod 1 \to M(*,x)$ requires at least that $M(*,x) \neq \emptyset$. When $\mathcal{C}$ is an enriched category with $\mathrm{ob}\,\mathcal{C} \neq \emptyset$ but whose hom objects $\mathcal{C}(x,x') = \emptyset$ for all $x,x' \in \mathrm{ob}\,\mathcal{C}$, an $(\mathcal{C},1)$-bimodule is at least a family of pointed sets $M(*,x)$. However, the family $\{A_x = \emptyset\}_{x\in\mathrm{ob}\,\mathcal{C}}$ of empty sets is an $\mathcal{C}$ algebra and thus not an $(\mathcal{C},1)$-bimodule.

The above counter-example suggests a bifurcation of roles for the terminal category that was not detectable in the ordinary 1-categorical case. In this latter, familiar landscape, $1$ simultaneously enjoys three important properties: it is the point classifier, it is terminal, and it has only "identity morphisms" on a single object. While we will not consider the first facet, we have just seen that in the general enriched case being terminal is not an indicator of having only "identity morphisms". Indeed, the extra structure on $(\mathcal{C}, 1)$-bimodules stems in a way from the fact that the terminal object in a monoidal category may nevertheless be "non-trivial" or have "internal structure".

Unfortunately there does not appear to be a general theory which suitably dictates how these roles are played and their connection to bimodules *qua* algebras. Thus, in absorbing the lesson of the above counter-example and with the absence of a suitable theory upon which to lean, we find ourselves adopting the approach of [Ste64, ¶14].

Recall that if a virtual double category has composites and units in the sense of [CS09, §5], then every horizontal identity $C \nrightarrow C$ is a monoid. In our case, $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$ has units and composites [CS09, A.10] which we leverage to define a certain enriched category which will play the role of defining algebras via bimodules.

**Definition B.2.4.** The **modularly simple** $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$**-category** $I$ has object set $\mathrm{ob}\, I = \{*\}$ and functor $I\colon \mathrm{Ind}_{\shortparallel}(\mathrm{ob}\, I) \to \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$ picking out the monoid $1 \nrightarrow 1$. ◁

To see that this is reasonable in this case, let us extract from Definition B.2.1 the notion of a left module $M$ for an enriched category $\mathcal{D}$ based at an object $C$: a family $M(y)\colon C \nrightarrow \mathcal{D}(y)$ acted on contravariantly by $\mathcal{D}$. Similarly we may specialise the notion of bimodule transformation to this case.

**Lemma B.2.5.** *The category of $C$-based left $I$-modules is isomorphic to $\mathscr{C}/(C \times T1)$.*

*Proof.* A left module for $I$ based at $C$ is the data of a span $C \leftarrow M \rightarrow T1$ as well as a morphism $\lambda_{*,*}\colon M \to M$ in $\mathscr{C}/(C \times T1)$. However, the unitality property of left modules with respect to the identity of the enriched category $I$ requires $\lambda_{*,*} = \mathrm{id}_M$. Thus each object of $\mathscr{C}/(C \times T1)$ has a unique left module structure. Finally, because all the structural maps of $I$ and its left modules are the identity, morphisms of left modules are precisely those of $\mathscr{C}/(C \times T1)$. ■

In this way, an $(\mathcal{C}, I)$-bimodule $M$ carries with it only the data of maps $M(x) \to T1$ in excess of the data of a right action of $\mathcal{C}$, and a compatibility condition between these. This serves as justification for the following definition.

**Definition B.2.6.** The **category of algebras** for a $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category $\mathcal{C}$ is defined to be the category $\mathrm{BiMod}(\mathcal{C}, I)$ of bimodules. ◁

We are finally equipped to compare algebras in this sense to algebras for the internalisation.

**Lemma B.2.7.** *Let $\mathcal{C}$ be a $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category, then there is a bimodule internalisation functor* $\mathrm{BiModInt}\colon \mathrm{BiMod}(\mathcal{C}, I) \to \mathrm{Alg}(\mathrm{Int}\,\mathcal{C})$ *following the structure of the functor* Int*, as well a an algebra externalisation functor* $\mathrm{AlgExt}\colon \mathrm{Alg}(\mathrm{Int}\,\mathcal{C}) \to \mathrm{BiMod}(\mathcal{C}, I)$ *fol-*

*lowing the structure of* Ext. *Moreover, there is an adjunction*

$$\begin{tikzcd} \mathrm{BiMod}(\mathcal{C}, I) \arrow[r, bend left, "\mathrm{BiModInt}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathrm{Int}\,\mathcal{C}) \arrow[l, bend left, "\mathrm{AlgExt}"] \end{tikzcd} ,$$

*whose co-unit is an isomorphism so that* $\mathrm{Alg}(\mathrm{Int}\,\mathcal{C})$ *is a reflective subcategory of* $\mathrm{BiMod}(\mathcal{C}, I)$.

*If* $T1$ *is terminal in* $\mathscr{C}$*, then these functors mediate an equivalence.*

*Proof.* An $(\mathcal{C}, I)$-bimodule $M$ comprises a family of spans $\{\mathcal{C}(x) \xleftarrow{m_x} M(x) \xrightarrow{p_x} T1\}$ as well as actions $\rho_{x,x'}$ rendering commutative the below diagram

$$\begin{tikzcd} & T1 & \\ M(x') \arrow[d, "m_{x'}"'] \arrow[ur, bend left, "p_{x'}"] & \mathcal{C}(x,x') \circ M(x) \arrow[l, "\rho_{x,x'}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TM(x) \arrow[d, "Tm_x"] \arrow[ul, bend right, "\mu_1(Tp_x)"'] \\ \mathcal{C}(x') & \mathcal{C}(x,x') \arrow[l, "f^{\ell}_{x,x'}"] \arrow[r, "f^{r}_{x,x'}"'] & T\mathcal{C}(x) \end{tikzcd} \tag{B.2.8}$$

,

and compatible with the composition and units of $\mathcal{C}$. By Lemma B.2.7, the category $I$ does not contribute further data or restrictions. A morphism $M \to N$ of $(\mathcal{C}, I)$-bimodules therefore involves a family of morphisms $\{\alpha_x \colon M(x) \to N(x)\}$ of spans, compatible with the action of $\mathcal{C}$ on $M$ and $N$.

On the other hand, an algebra for $\mathrm{Int}\,\mathcal{C}$ comprises a sole object $M$ and morphism $m \colon M \to \coprod_{x \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x)$, as well as a map $\rho$ rendering commutative the below diagram

$$\begin{tikzcd} M \arrow[d, "m"'] & (\coprod_{x,x' \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x,x')) \circ M \arrow[l, "\rho"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TM \arrow[d, "Tm"] \\ \coprod_{x \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x') & \coprod_{x,x' \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x,x') \arrow[l, "{[\iota_{x'} f^{\ell}_{x,x'}]}"] \arrow[r, "{[(T\iota_x) f^{r}_{x,x'}]}"'] & T \coprod_{x \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x) \end{tikzcd} \tag{B.2.9}$$

,

and compatible with the composition and units of $\mathrm{Int}\,\mathcal{C}$. A morphism of $M \to N$ of algebras therefore involves a morphism $\alpha \colon M \to N$ compatible with the action of $\mathrm{Int}\,\mathcal{C}$ on $M$ and $N$.

However, we can simplify matters further. For such an algebra, let us write $m_x \colon M(x) \to \mathcal{C}(x)$ for the left leg of the below pullback.

$$\begin{tikzcd} M(x) \arrow[r] \arrow[d, "m_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "m"] \\ \mathcal{C}(x) \arrow[r, "\iota_x"'] & \coprod_{\mathrm{ob}\,\mathcal{C}} \mathcal{C}(x) \end{tikzcd}$$

Because $\mathscr{C}$ is infinitary extensive we have $M = \coprod_{\mathrm{ob}\,\mathcal{C}} M_x$ and $m = \coprod m_x$ which allows us to rewrite the pullback (B.2.9) above as a coproduct of pullbacks, so that in particular its vertex is $\coprod_{x,x' \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x,x') \circ M(x)$. Similarly therefore $\rho = [\iota_{x'} \rho_{x,x'}]$.

By this comparison it is hoped that the proof suggests itself to the reader. Given an $(\mathcal{C}, I)$-bimodule as above, we may discard the data $\{M(x) \to T1\}$ and form precisely the structure of an $\operatorname{Int}\mathcal{C}$ algebra. In this way, we obtain a bimodule internalisation functor $\mathrm{BiModInt}\colon \mathrm{BiMod}(\mathcal{C}, I) \to \mathrm{Alg}(\operatorname{Int}\mathcal{C})$ which will serve as our left adjoint.

The right adjoint is somewhat more involved, but once the basic data is fixed the rest follows mechanically. Beginning this time with $M = \coprod_{x \in \operatorname{ob}\mathcal{C}} M(x)$ an $\operatorname{Int}\mathcal{C}$ algebra, we may obtain a family of spans $\{\mathcal{C}(x) \leftarrow M(x) \times T1 \to T1\}$ by freely multiplying by $T1$. Some amount of pullback yoga is required, but inevitably the data $\rho = [\iota_{x'}\rho_{x,x'}]$ allows us to supply the spans $\mathcal{C}(x) \leftarrow M(x) \times T1 \to T1$ with actions by the $\mathcal{C}(x, x')$ by factoring the required map as a pair of actions: $\rho_{x,x'}$ on $M(x)$ and $\mu$ on $T1$. The resulting diagrams (B.2.8) are commutative precisely because of this free separation of actions. All of these considerations give rise to the algebra externalisation functor, $\mathrm{AlgExt}\colon \mathrm{Alg}(\operatorname{Int}\mathcal{C}) \to \mathrm{BiMod}(\mathcal{C}, I)$.

As is usual for freely multiplying, we may observe that bimodule transformations $N \to \mathrm{AlgExt}\, M$ are uniquely determined by the data of morphisms $N(x) \to M(x)$ over each $\mathcal{C}(x)$ alone. In turn this data is the same as that describing an algebra morphism $\mathrm{BiModInt}\, N \to M$, whence the adjunction. Let us observe that the counit is an isomorphism: we have already determined that each algebra $M$ for $\operatorname{Int}\mathcal{C}$ may be rewritten as a coproduct, and $\mathrm{BiModInt}\,\mathrm{AlgExt}\, M$ gives precisely this rewriting.

To conclude the proof we note that in (B.2.8), if $T1$ is terminal then the unit is an isomorphism for it has components $\langle p_x, M(x)\rangle\colon M(x) \to M(x) \times T1$ where $p_x\colon M(x) \to T1$ is uniquely determined. ■

## B.3. From $T$-algebras to $T$-spans

Our particular application involves another coincidence of the theory of $T$-categories: there is a canonical functor from the category of $T$-algebras to $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$.

**Lemma B.3.1.** *Consider the monoidal category $\mathscr{C}^T$ as a degenerate virtual double category. The assignment of horizontal arrows*

$$\big((A, h)\colon * \nrightarrow *\big) \mapsto \Big(\big(1 \leftarrow A \times T1 \xrightarrow{\pi} T1\big)\colon 1 \nrightarrow 1\Big)$$

*extends to a functor $R\colon \mathscr{C}^T \to \mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$.*

However, while we could prove this at the level of virtual double categories we can make more efficient our proof by noticing that $R$ is determined by a lax monoidal functor. To make sense of this we will need to factor the image of $R$ as the inclusion of a monoidal category into $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$.

Recall that in a bi-category, each endo-hom-category is a monoidal category. So too an analogous phenomenon occurs when the virtual double category at hand has composites and units, one may observe that it is possible to extract from such a virtual double category its monoidal categories of horizontal endomorphisms. For expediency we shall not describe this phenomenon in general, and instead give a direct account of the monoidal category we intend to extract from $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$.

**Definition B.3.2.** The **monoidal category** $(\mathscr{C}/T1, \square, \eta_1)$ **of collections** is defined to be the monoidal category $\mathrm{Span}(\mathscr{C}, T)(1,1)$, endomorphisms of 1 in the bi-category of $T$-spans, where we have identified $1 \times T1 \cong T1$. In detail, the monoidal unit is $\eta_1 \colon 1 \to T1$ and the **composition tensor product on** $\mathscr{C}/T1$, applied to of a pair of objects $C = (C, C^r)$ and $D = (D, D^r)$ of $\mathscr{C}/T1$ is defined to be the top composite in the below diagram. To avoid confusion we write this tensor product with a box.

$$\begin{tikzcd} C \square D \arrow[r, "p_D"] \arrow[d, "p_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & TD \arrow[r, "TD^r"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1"] & T1 \\ C \arrow[r, "C^r"'] & T1 \end{tikzcd}$$

Note that the monoidal category of collections appears as the full sub-virtual double category of $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$ on the object 1. ◁

Now we are equipped to give a proof of our lemma above.

*Proof* (Lemma B.3.1). We shall construct instead a lax monoidal functor

$$(R, \varphi) \colon (\mathscr{C}^T, \times, 1) \to (\mathscr{C}/T1, \square, \eta_1)$$

which may therefore be recast as a virtual double functor from $\mathscr{C}^T$ into a (sub-virtual double category) of $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$. To that end, notice that the functor $R$ about which we wish to reason is realised as the following composite

$$\mathscr{C}^T \xrightarrow{U^T} \mathscr{C} \cong \mathscr{C}/1 \xrightarrow{(!_{T1})^*} \mathscr{C}/T1 \;.$$

First let us dispense with the unitor. Note that $R(1, !_{T1})$ is terminal in $\mathscr{C}/T1$ so that there is a unique morphism $\phi \colon \eta_1 \to R(1, !_{T1})$. Next, take objects $(A, h)$ and $(B, g)$ and let us define $\phi_{(A,h),(B,g)} \colon R(A,h) \square R(B,g) \to R\big((A,h) \times (B,g)\big)$ as follows. Note that $R(A,h) \square R(B,g)$ is given by the top composite in the below pullback.

$$\begin{tikzcd} A \times T(B \times T1) \arrow[r, "\pi"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(B \times T1) \arrow[r, "T\pi"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1"] & T1 \\ A \times T1 \arrow[r, "\pi"'] & T1 \end{tikzcd}$$

Recall that $T$ is cartesian so that $T(B \times T1) = TB \times_{T1} T^2 1$. Therefore there is a canonical morphism $u_{A,B} \colon R(A,h) \square R(B,g) \to (A \times TB) \times T^2 1$ in $\mathscr{C}/T1$ where the codomain is considered as an object of $\mathscr{C}/T1$ by the morphism $\mu_1 \pi$. We therefore define $\varphi_{(A,h),(B,h)}$ to be the top composite of the below diagram. That $\varphi$ is natural may be checked in parts, as indicated by the below commutative diagram.

$$\begin{tikzcd} R(A,h)\square R(B,g) \arrow[r, "u_{A,B}"] \arrow[d, "Ra\square Rb"'] & (A\times TB)\times T^2 1 \arrow[r, "(\mathrm{id}_A\times g)\times\mu_1"] \arrow[d, dashed, "(a\times Tb)\times\mathrm{id}_{T^2 1}"] \arrow[dl, phantom, "\text{univ. prop.}"] \arrow[dr, phantom, "\text{alg. morph.}"] & (A\times B)\times T1 \arrow[d, "(a\times b)\times\mathrm{id}_{T1}"] \\ R(A',h')\square R(B',g') \arrow[r, "u_{A',B'}"'] & (A'\times TB')\times T^2 1 \arrow[r, "(\mathrm{id}_{A'}\times g')\times\mu_1"'] & (A'\times B')\times T1 \end{tikzcd}$$

Finally we must verify that the various laws governing associators and unitors hold, but these are all straightforward universal property arguments. Pleasingly, the reason that the associativity law holds is essentially due to the fact that algebra maps $h\colon TA \to A$ are associative, and similarly the reason that the unit laws hold is because algebra maps are unital. ■

It is evident from the above description that $R$ has a left adjoint *functor*, because it is a composite of right adjoints. It is moreover true by doctrinal adjunction [Kel74] that its left adjoint functor will be at least op-lax monoidal in strength. However, as is straightforwardly verified the left adjoint functor to $R$ is not strong monoidal (equivalently lax monoidal) and therefore the left adjoint is entirely absent at the level of virtual double categories for it does not map cells to cells. We record this curiosity here, but have no further use for this observation.

**Observation B.3.3.** *The lax monoidal functor $R\colon \mathscr{C}^T \to \mathscr{C}/T1$ has a monoidal left adjoint given by the composite*

$$\mathscr{C}/T1 \xrightarrow{\Sigma_{!_{T1}}} \mathscr{C}/1 \cong \mathscr{C} \xrightarrow{F^T} \mathscr{C}^T \ ,$$

*but the left adjoint is only op-lax monoidal.* ■

Having decided upon a suitable functor $R\colon \mathscr{C}^T \to \mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$ we are interested in comparing algebras under the action of $R$. To our embarrassment, once more we find ourselves without direction from more general considerations in determining which $\mathscr{C}^T$-category trivialises appropriately the contravariant action on bimodules. Thus, in what we hope is the final instance, we appeal to the work of [Ste64] to justify the following definition.

**Definition B.3.4.** The **modularly simple $\mathscr{C}^T$-category $J$** has object set $\mathrm{ob}\,J = \{*\}$ and functor $J\colon \mathrm{Ind}_{\mathbb{H}}(\mathrm{ob}\,J) \to \mathscr{C}^T$ picking out the monoid $(1, !\colon T1 \to 1)\colon * \nrightarrow *$.

The **category of algebras for a $\mathscr{C}^T$-category** $\mathcal{C}$ is defined to be $\mathrm{BiMod}(\mathcal{C}, J)$. ◂

*Remark* B.3.5. Here, unlike in the case of Definition B.2.4, there is a coincidence between the “monoidal unit” and the “terminal object”. Thus, $J$ is in fact the terminal $\mathscr{C}^T$-category whereas $I$ is not the terminal $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)$-category. ◂

As was the case in Definition B.2.4 and Lemma B.2.5 we may inspect $*$-based left $J$-modules to assure ourselves that this definition is appropriate.

**Lemma B.3.6.** *The category of $*$-based left $J$-modules is isomorphic to $\mathscr{C}^T$.* ■

Now, by Lemmas B.3.1 and B.2.2, there is a functor

$$\mathrm{BiMod}_R \colon \mathrm{BiMod}_{\mathscr{C}^T}(\mathcal{C}, J) \to \mathrm{BiMod}_{\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)}(R_*\mathcal{C}, R_*J)$$

for every $\mathscr{C}^T$-category $\mathcal{C}$. However, inasmuch as algebras are concerned we are not interested in $R_*J$ but rather $I$ of Definition B.2.4. Fortunately there is an evident $\mathbb{H}$-Kl($\mathbb{S}$pan$\,\mathscr{C}, T$)-functor.

**Lemma B.3.7.** *There is a canonical functor $\varphi \colon I \to R_*J$ of* $\mathbb{H}$-Kl($\mathbb{S}$pan$\,\mathscr{C}, T$)*-categories.*

*Proof.* Both $\mathbb{H}$-Kl($\mathbb{S}$pan$\,\mathscr{C}, T$)-categories have a single object and so this amounts to giving a morphism of monoids $(I \nrightarrow I) \to R_*J$ in the virtual double category $\mathbb{H}$-Kl($\mathbb{S}$pan$\,\mathscr{C}, T$). However, $R_*J$ is the terminal endo-$T$-span $1 \leftarrow T1 = T1$ and so the result follows. Alternatively one may see this $\mathbb{H}$-Kl($\mathbb{S}$pan$\,\mathscr{C}, T$)-functor as arising from the fact that the unitor $\varphi \colon I \to RJ$ of a lax monoidal functor $(R, \varphi)$ is a morphism of monoids. ∎

By using this functor, and appealing to Lemmas B.2.2 and B.2.7 we may therefore restrict $(R_*\mathcal{C}, R_*J)$-bimodules to $(R_*\mathcal{C}, I)$-bimodules. At first glance it would appear that such a functor, $\mathrm{BiMod}(\mathrm{id}, \varphi) \circ \mathrm{BiMod}_R$ ought to be a poor comparison of algebras. Indeed, $R$ and consequently $\mathrm{BiMod}_R$ appear to forget a great deal about the objects of $\mathscr{C}^T$—the $T$-algebra structure on the carrier objects would seem to be *oublié*. Surprisingly however, the structure of being an $R_*\mathcal{C}$ algebra is such that inescapably such objects carry with them enough information to recover the $T$-algebra structure on their carriers.

In Appendix B.2 we saw that Int did not generally admit an adjoint but the induced comparison on algebras did. Now we shall be twice lucky for the same fact, in the context of $R$ instead.

**Lemma B.3.8.** *Let $\mathcal{C}$ be a $\mathscr{C}^T$-category and let $M$ in* $\mathrm{BiMod}_{\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)}(R_*\mathcal{C}, I)$ *be an algebra with underlying data $\{p_x \colon M(x) \to T1\}_{x \in \mathrm{ob}\,\mathcal{C}}$.*

*Each object $M(x)$ canonically carries the structure of a $T$-algebra. Using this, there is a functor*[13] $\mathrm{Réc} \colon \mathrm{BiMod}_{\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)}(R_*\mathcal{C}, I) \to \mathrm{BiMod}_{\mathscr{C}^T}(\mathcal{C}, J)$ *which recovers these algebra structures so that in particular the carrier object for each* $(\mathrm{Réc}\, M)(x) \in \mathscr{C}^T$ *is $M(x)$.*

*Proof.* Let us tend to the first claim, for it is a great portion of the proof of the second. Let $M$ in $\mathrm{BiMod}_{\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T)}(R_*\mathcal{C}, I)$ be an algebra, so that we may write as above $\{p_x \colon M(x) \to T1\}_{x \in \mathrm{ob}\,\mathcal{C}}$ for the family of objects of $\mathscr{C}/T1$. In addition we have a family of maps $\{\rho_{x,y} \colon \mathcal{C}(x,y) \times TM(x) \to M(y)\}_{x,y \in \mathrm{ob}\,\mathcal{C}}$ which render commutative the below left diagram. Our challenge is to extract from these a suitable map $h(x) \colon TM(x) \to M(x)$. For this purpose let us define $h(x)$ to be the morphism represented by the below-right cell in $\mathbb{H}$-Kl($\mathbb{S}$pan$\,\mathscr{C}, T$), which we shall shortly elaborate.

[13] Réc for *récupérer*

$$\begin{tikzcd}
\mathcal{C}(x,y)\times TM(x) \arrow[r, "\rho_{x,y}"] \arrow[d, "\pi"'] & M(y) \arrow[ddl, "p_x"] \\
TM(x) \arrow[d, "\mu_1(Tp_x)"'] & \\
T1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "R1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "R\,\mathsf{refl}_x"] & 1 \arrow[r, "M(x)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_{M(x)}"] & 1 \arrow[d, equal] \\
1 \arrow[r, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho_{x,x}"] & 1 \arrow[r, "M(x)"', "\shortmid"{marking}] & 1 \arrow[d, equal] \\
1 \arrow[rr, "M(x)"', "\shortmid"{marking}] & & 1
\end{tikzcd}
\tag{B.3.9}$$

Should we expand the above-right cell composite in $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$ we see that we have defined $h(x)$ as follows.

$$h(x) :\equiv \left(TM(x) \xrightarrow{\langle \mathsf{refl}_x, \mathrm{id}_{TM(x)}\rangle} \mathcal{C}(x,x)\times TM(x) \xrightarrow{\rho_{x,x}} M(x)\right) \tag{B.3.10}$$

In understanding what $h(x)$ was defined to be, it is also important to understand what it was not defined to be. In making the definition above we have taken advantage of the fact that the arrow $\mathsf{refl}_{\mathcal{C}}\colon 1 \to \mathcal{C}(x,x)$ of the 1-category $\mathscr{C}^T$ is both a nullary cell $* \to \mathcal{C}(x,x)$ and a unary cell $1 \to \mathcal{C}(x,x)$ in the virtual double category $\mathscr{C}^T$ for 1 is the monoidal unit. With this in mind, below-left we have produced one form of the unitality law for $\rho_{x,x}$ in $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$, and expanded the cell composite to $\mathbb{S}\mathrm{pan}\,\mathscr{C}$ below-right. By examining this we see that $h(x)\eta_{M(x)} = \mathrm{id}_{M(x)}$ for $\rho$ is unital.

$$\begin{tikzcd}
& 1 \arrow[dr, equal] \arrow[d, phantom, "R\,\mathsf{refl}_x"] & \\
1 \arrow[ur, equal] \arrow[r, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho_{x,x}"] & 1 \arrow[r, "M(x)"', "\shortmid"{marking}] & 1 \arrow[d, equal] \\
1 \arrow[rr, "M(x)"', "\shortmid"{marking}] & & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dr, bend left, "\eta_1"] \arrow[d, phantom, "R\,\mathsf{refl}_x"] & & \\
1 \arrow[ur, equal] \arrow[rr, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\rho_{x,x}"] & & T1 \arrow[r, "TM(x)"', "\shortmid"{marking}] & T^21 \arrow[d, "\mu_1"] \\
1 \arrow[rrr, "M(x)"', "\shortmid"{marking}] & & & T1
\end{tikzcd}$$

$\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$ $\qquad\qquad$ $\mathbb{S}\mathrm{pan}\,\mathscr{C}$

To conclude the claim that $(M(x),h(x))$ is a $T$-algebra we must prove associativity. Pleasingly this claim may be deduced from the associativity laws of $\rho$ as we now observe. We have produced below the cell equality satisfied by $\rho$ in $\mathbb{H}\text{-Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)$, pre-composed by $R\,\mathsf{refl}_x$, and elaborated in $\mathbb{S}\mathrm{pan}\,\mathscr{C}$.

$$\begin{tikzcd}
1 \arrow[r, "R1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "R\,\mathsf{refl}_x"] & T1 \arrow[r, "TR1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "R\,\mathsf{refl}_x"] & T^21 \arrow[r, "T^2M(x)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "T^2\,\mathrm{id}_{M(x)}"] & T^31 \arrow[d, equal] \\
1 \arrow[r, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "R\,\mathsf{comp}_{x,x,x}"] & T1 \arrow[r, "TR(x,x)"', "\shortmid"{marking}] & T^21 \arrow[r, "T^2M(x)"', "\shortmid"{marking}] \arrow[d, "\mu_1"] \arrow[dr, phantom, "\mu_{M(x)}"] & T^31 \arrow[d, "\mu_{T1}"] \\
1 \arrow[rr, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\rho_{x,x}"] & & T1 \arrow[r, "TM(x)"', "\shortmid"{marking}] & T^21 \arrow[d, "\mu_1"] \\
1 \arrow[rrr, "M(x)"', "\shortmid"{marking}] & & & T1
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r, "R1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "R\,\mathsf{refl}_x"] & T1 \arrow[r, "TR1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "R\,\mathsf{refl}_x"] & T^21 \arrow[r, "T^2M(x)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "T^2\,\mathrm{id}_{M(x)}"] & T^31 \arrow[d, equal] \\
1 \arrow[r, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_{R\mathcal{C}(x,x)}"] & T1 \arrow[r, "TR(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "T\rho_{x,x}"] & T^21 \arrow[r, "T^2M(x)"', "\shortmid"{marking}] & T^31 \arrow[d, "\mu_{T1}"] \\
1 \arrow[r, "R\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\rho_{x,x}"] & T1 \arrow[rr, "TM(x)"', "\shortmid"{marking}] & & T^21 \arrow[d, "\mu_1"] \\
1 \arrow[rrr, "M(x)"', "\shortmid"{marking}] & & & T1
\end{tikzcd}$$

By elaborating the above-left diagram, and making use of some pullback lemmas and the unitality of $\mathsf{comp}_{x,x,x} \circ \langle \mathrm{id}, \mathsf{refl}_x \rangle = \mathrm{id}$, we see that it presents the cell

$$T^2 M(x) \xrightarrow{\mu_{M(x)}} TM(x) \xrightarrow{h(x)} M(x) \ .$$

A more straightforward computation for the above-right diagram gives that the presented cell is $h(x)(Th(x))$, whence $(M(x), h(x))$ is a $T$-algebra. At this point we may note that an added compatibility. Because morphisms of bimodules respect the right action of $R_*\mathcal{C}$ from which the algebra structure is derived, one may verify that every morphism $\{M(x) \to M'(x)\}_{x \in \mathrm{ob}\, \mathcal{C}}$ of $(R_*\mathcal{C}, I)$-bimodules generates a family of morphisms $\{(M(x), h(x)) \to (M'(x), h'(x))\}_{x \in \mathrm{ob}\, \mathcal{C}}$ of $T$-algebras in a functorial manner.

Now let us turn our attention to describing the functor Réc. Given a $(R_*\mathcal{C}, I)$-bimodule $M$ we have just constructed a family of $T$-algebras $\{(M(x), h(x))\}_{x \in \mathrm{ob}\, \mathcal{C}}$. We will use these, as well as the extant right action $\rho$ of $R_*\mathcal{C}$ on $M$ to give a new action $\{\mathrm{Réc}\, \rho_{x,y} \colon \mathcal{C}(x,y) \times M(x) \to M(y)\}_{x,y \in \mathrm{ob}\, \mathcal{C}}$ and prove that the result is an $\mathcal{C}$ algebra.

To that end let us define $\mathrm{Réc}\, \rho$ to be the composite

$$\mathrm{Réc}\, \rho_{x,y} :\equiv \left( \mathcal{C}(x,y) \times M(x) \xrightarrow{\mathrm{id} \times \eta_{M(x)}} \mathcal{C}(x,y) \times TM(x) \xrightarrow{\rho_{x,y}} M(y) \right) \ ,$$

which is presented by the following cell composite in $\mathbb{S}\mathrm{pan}\,\mathscr{C}$.

$$\begin{tikzcd}
 & M(x) \arrow[dl, bend right, "\eta!"'] \arrow[dr, bend left, "\eta p_x"] \arrow[d, phantom, "\eta"] & \\
1 \arrow[r, "R\mathcal{C}(x,y)"', "|" marking] \arrow[d, equal] & T1 \arrow[r, "TM(x)"', "|" marking] \arrow[d, phantom, "\rho_{x,y}"] & T^2 1 \arrow[d, "\mu_1"] \\
1 \arrow[rr, "M(y)"', "|" marking] & & T1
\end{tikzcd}$$

To establish that this is a valid $\mathcal{C}$ algebra structure we must show that associativity and unitality are obeyed, as well as demonstrate that each component is a morphism of $T$-algebras. It is hoped from our presentation of $\mathrm{Réc}\, \rho$ as a certain cell composite in $\mathbb{S}\mathrm{pan}\,\mathscr{C}$ that proof of these properties suggests itself to the reader. Indeed, associativity and unitality follow directly from those same properties of $\rho$. The fact that $\mathrm{Réc}\, \rho$ is a morphism of algebras follows from the fact that $h(x)$ was itself defined in terms of $\rho$, that $\rho$ is associative, and that the functor $R$ introduces the $T$-algebra structural morphisms $h(x,y) \colon T\mathcal{C}(x,y) \to \mathcal{C}(x,y)$ into cell compositions in $\mathbb{S}\mathrm{pan}\,\mathscr{C}$.

To conclude the proof we must determine that $(R_*\mathcal{C}, I)$-bimodule morphisms, which we have already determined to functorially descend to morphisms of the $T$-algebras $(M(x), h(x))$, interact appropriately with $\mathrm{Réc}\, \rho$ above. Once again it is hoped that the cell diagram above suggests the proof: this follows precisely because $(R_*\mathcal{C}, I)$-bimodule morphisms interact appropriately with the action of $R_*\mathcal{C}$. In this way we have determined the functor Réc. ■

**Lemma B.3.11.** *Let us write* Oub [14] *for the following functor on categories of algebras.*

$$\mathrm{Oub} :\equiv \mathrm{BiMod}(\mathrm{id}, \varphi) \circ \mathrm{BiMod}_R \colon \mathrm{BiMod}_{\mathscr{C}^T}(\mathcal{C}, J) \to \mathrm{BiMod}_{\text{H-Kl}(\mathrm{Span}\,\mathscr{C}, T)}(R_*\mathcal{C}, I)$$

Oub *is embroiled in an adjunction* Réc ⊣ Oub *and both functors are faithful. If* $T1 = 1$ *then these functors mediate an equivalence.*

*Proof.* For the first part see for example [Mac71, Thm. IV.3.1]. For the second part observe that any such global element gives a section $\langle \mathrm{id}_A, b! \rangle \colon A \to A \times B$ of the projection. ■

Now we may proceed with the proof of Lemma B.3.11.

*Proof* (Lemma B.3.11). We shall begin by describing the co-unit $\varepsilon \colon \mathrm{Réc}\,\mathrm{Oub} \Rightarrow \mathrm{id}$ of this adjunction and demonstrate that it has the appropriate universal property. To that end, fix an $\mathcal{C}$ algebra $M$ so that we may compute $\mathrm{Réc}\,\mathrm{Oub}\,M$ to comprise the data of $T$-algebras on $M(x) \times T1$ as below-left and actions of $\mathcal{C}(x,y)$ on these as below-right.

$$\begin{tikzcd}
T(M(x) \times T1) \arrow[d, "\langle T\pi, T\pi \rangle"'] \\
TM(x) \times T^2 1 \arrow[d, "h(x) \times \mu_1"'] \\
M(x) \times T1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(x,y) \times (M(x) \times T1) \arrow[d, "\text{assoc.}"] \\
(\mathcal{C}(x,y) \times M(x)) \times T1 \arrow[d, "\rho_{x,y} \times \mathrm{id}"] \\
M(y) \times T1
\end{tikzcd}$$

From this description we see that the evident morphisms $\pi \colon M(x) \times T1 \to M(x)$ of $\mathscr{C}$ are in fact morphism of $T$-algebras. Moreover, as may be readily verified, these morphisms extend to morphisms of $\mathcal{C}$ algebras $\varepsilon_M \colon \mathrm{Réc}\,\mathrm{Oub}\,M \to M$. Naturality is a straightforward computation, and so to complete the proof of the adjunction we must attend to the universal property.

Given an $(\mathcal{C}, J)$-bimodule transformation $\{\alpha_x \colon (\mathrm{Réc}\,M')(x) \to M(x)\}_{x \in \mathrm{ob}\,\mathcal{C}}$, we must show that there is a unique $(R_*\mathcal{C}, I)$-bimodule transformation $\overline{\alpha} \colon M' \to \mathrm{Oub}\,M$ such that $\varepsilon_M \overline{\alpha} = \alpha$. Let us dispense with uniqueness first.

Observe that a bimodule transformation $\overline{\alpha} \colon M' \to \mathrm{Oub}\,M$ comprises the data of a family $\{\overline{\alpha}_x \colon M'(x) \to M(x) \times T1\}_{x \in \mathrm{ob}\,\mathcal{C}}$ such that $\pi_{T1}\overline{\alpha}_x = p'_x$ where the legs $\{p'_x \colon M'(x) \to T1\}_{x \in \mathrm{ob}\,\mathcal{C}}$ are the data of $M'$. Thus, in requiring additionally that $\varepsilon_M \overline{\alpha} = \alpha$ we thereby fix $\overline{\alpha}_x = \left\langle \alpha_x, p'_x \right\rangle$ uniquely.

Now we must show existence, which is reduced to the claim that the family $\overline{\alpha}$ just given is a transformation of $(R_*\mathcal{C}, I)$-bimodules. In turn this requires that we demonstrate the following cell composition equality in $\mathrm{Span}\,\mathscr{C}$.

[14] for *oublier*

$$
\begin{tikzcd}
1 \arrow[r, "R\mathcal{C}(x,y)", "\shortmid" marking] \arrow[d, equal] & T1 \arrow[r, "TM'(x)", "\shortmid" marking] \arrow[d, phantom, "\rho'_{x,y}"] & T^2 1 \arrow[d, equal] \\
1 \arrow[rr, "M'(y)"', "\shortmid" marking] \arrow[d, equal] & \arrow[d, phantom, "\overline{\alpha}_y"] & T1 \arrow[d, "\mu_1"] \\
1 \arrow[rr, "RM(y)"', "\shortmid" marking] & & T1
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r, "R\mathcal{C}(x,y)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}"] & T1 \arrow[r, "TM'(x)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "T\overline{\alpha}_y"] & T^2 1 \arrow[d, equal] \\
1 \arrow[r, "R\mathcal{C}(x,y)"', "\shortmid" marking] \arrow[d, equal] & T1 \arrow[r, "TRM(x)"', "\shortmid" marking] \arrow[d, phantom, "R\rho_{x,y}"] & T^2 1 \arrow[d, "\mu_1"] \\
1 \arrow[rr, "RM(y)"', "\shortmid" marking] & & T1
\end{tikzcd}
\tag{B.3.12}
$$

Our assumption that $\alpha\colon \operatorname{Réc} M' \to M$ is a transformation of $(\mathcal{C}, J)$-bimodules gives us the commutativity of the below diagram in $\mathscr{C}^T$ which we will now leverage to establish the above equality.

$$
\begin{tikzcd}
\mathcal{C}(x,y) \times M'(x,y) \arrow[r, "\mathrm{id} \times \eta"] \arrow[d, "\mathrm{id} \times \alpha_x"'] & \mathcal{C}(x,y) \times TM'(x,y) \arrow[r, "\rho'_{x,y}"] & M'(y) \arrow[d, "\alpha_y"] \\
\mathcal{C}(x,y) \times M(x) \arrow[rr, "\rho_{x,y}"'] & & M(y)
\end{tikzcd}
\tag{B.3.13}
$$

By unwinding the cell composition exhibited in B.3.12 and noting that the eventual codomain is a product $M(y) \times T1$, we see that we are asking for the commutativity of the following two diagrams.

$$
\begin{tikzcd}
\mathcal{C}(x,y) \times TM'(x) \arrow[r, "\rho'_{x,y}"] \arrow[d, "\pi"'] & M'(y) \arrow[d, "p'_y"] \\
TM'(x) \arrow[r, "\mu_1(Tp'_x)"'] & T1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(x,y) \times TM'(x) \arrow[r, "\rho'_{x,y}"] \arrow[d, "\mathrm{id} \times h'(x)"'] & M'(y) \arrow[d, "\alpha_y"] \\
\mathcal{C}(x,y) \times M'(x) \arrow[r, "\rho_{x,y}(\mathrm{id} \times \alpha_x)"'] & M(y)
\end{tikzcd}
$$

In the above-right diagram we have made use of the simplification $\alpha_x$ is a morphism of $\mathscr{C}^T$ where we have written $h'(x)$ for $T$-algebra structure may on $M'(x)$ given by $(\operatorname{Réc} M')(x)$.

The above-left diagram is commutative because it is precisely the description of the boundary of the cell $\rho'_{x,y}$ in $\mathbb{S}\mathrm{pan}\,\mathscr{C}$. To demonstrate that the above-right diagram is commutative requires some extra care. First we may appeal to the commutativity of (B.3.13) and the naturality of $\eta$ to reduce the claim to the equality

$$
\alpha_y \rho'_{x,y} = \alpha_y \rho'_{x,y} \Big( \mathrm{id} \times \big( (Th'(x)) \eta_{M'(x)} \big) \Big) .
$$

Recall from Lemma B.3.8 that $h'(x)$ is defined as in (B.3.10) and diagrammatically as in (B.3.9). With this, we may reduce the claim of the above equation the following argument on cells of $\mathbb{S}\mathrm{pan}\,\mathscr{C}$, where we have omitted post-composition by $\alpha_y$ because the equality happens earlier.

$$
\begin{tikzcd}
 & & & TM'(x) \arrow[dl, "\eta T!"'] \arrow[dr, "\eta T p'_x"] \arrow[d, phantom, "\eta"] & \\
1 \arrow[r, "R\mathcal{C}(x,y)", "\shortmid"{marking}] \arrow[dd, equal] \arrow[dr, phantom, "\mathrm{id}"] & T1 \arrow[r, "TR1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "TR\,\mathsf{refl}_x"] & T^21 \arrow[rr, "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mathrm{id}"] & & T^31 \arrow[d, equal] \\
 & T1 \arrow[r, "TR\mathcal{C}(x,x)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "T\rho'_{x,x}"] & T1 \arrow[rr, "T^2M'(x)"', "\shortmid"{marking}] & & T^31 \arrow[d, "\mu_{T1}"] \\
1 \arrow[r, "R\mathcal{C}(x,y)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrrr, phantom, "\rho'_{x,y}"] & T1 \arrow[rrr, "TM'(x)"', "\shortmid"{marking}] & & & T^21 \arrow[d, "\mu_1"] \\
1 \arrow[rrrr, "M'(y)"', "\shortmid"{marking}] & & & & T1
\end{tikzcd}
\overset{\text{assoc.}}{=}
\begin{tikzcd}
 & & & TM'(x) \arrow[dl, "\eta T!"'] \arrow[dr, "\eta T p'_x"] \arrow[d, phantom, "\eta"] & \\
1 \arrow[r, "R(x,y)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}"] & T1 \arrow[r, "TRI", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "TR\,\mathsf{refl}_x"] & T^21 \arrow[rr, "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\mu"] & & T^31 \arrow[dd, "\mu_{T1}"] \\
1 \arrow[r, "R\mathcal{C}(x,y)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "R\,\mathsf{comp}_{x,x,y}"] & T1 \arrow[r, "TR(x,x)"', "\shortmid"{marking}] & T^21 \arrow[d, "\mu_1"] & & \\
1 \arrow[rr, "R\mathcal{C}(x,y)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrrr, phantom, "\rho'_{x,y}"] & & T1 \arrow[rr, "TM'(x)"', "\shortmid"{marking}] & & T^21 \arrow[d, "\mu_1"] \\
1 \arrow[rrrr, "M'(y)"', "\shortmid"{marking}] & & & & T1
\end{tikzcd}
$$

$$
\overset{\text{unit.}}{=}
\begin{tikzcd}
1 \arrow[r, "R\mathcal{C}(x,y)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho'_{x,y}"] & T1 \arrow[r, "TM'(x)", "\shortmid"{marking}] & T^21 \arrow[d, "\mu_1"] \\
1 \arrow[rr, "M'(y)"', "\shortmid"{marking}] & & T1
\end{tikzcd}
$$

This concludes the existence of a factorisation through $\varepsilon$, and overall the proof that Réc $\vdash$ Oub.

To conclude the proof let us demonstrate that both functors are faithful. This occurs precisely when the co-unit is an epimorphism and the unit is a monomorphism. Our co-unit is given by projections $\{M(x) \times T1 \to M(x)\}$, but $\eta_1 \colon 1 \to T1$ is a global element and therefore a splitting of the projection. Thus the co-unit is split epi, and so Oub is a faithful. Finally we may compute the unit to be given by the family $\{\langle \mathrm{id}, p_x \rangle \colon M(x) \to M(x) \times T1\}_{x \in \mathrm{ob}\, \mathcal{C}}$ in $\mathscr{C}/T1$ so that $\eta$ is a split monomorphism and Réc is faithful.

To conclude the proof it remains to observe that if $T1 = 1$ both the unit and the co-unit are isomorphisms so that Oub and Réc mediate an equivalence. ■

There is a final, fortunate coincidence of algebraic structures at play here. Should we inspect the definitions of Oub and that of AlgExt we may not that the produce the same sorts of bimodules in $\mathrm{BiMod}_{\mathrm{H\text{-}Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)}(R_*\mathcal{C}, I)$, and the same sort of transformations between these. In fact, by inspection we may deduce the following.

**Lemma B.3.14.** *The image of* $\mathrm{Oub} \colon \mathrm{BiMod}_{\mathscr{C}^T}(\mathcal{C}, J) \to \mathrm{BiMod}_{\mathrm{H\text{-}Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)}(R_*\mathcal{C}, I)$ *is the same as the image of* $\mathrm{AlgExt} \colon \mathrm{Alg}(\mathrm{Int}\, R_*\mathcal{C}) \to \mathrm{BiMod}_{\mathrm{H\text{-}Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C},T)}(R_*\mathcal{C}, I)$. ■

To end the section, let us combine all of the results we have obtained so far to give the promised comparison of algebras of $\mathcal{C}$, a $\mathscr{C}^T$-category, to those of $\mathrm{Int}(R_*\mathcal{C})$.

**Theorem B.3.15** (Grothendieck construction)**.** *The composite*

$$\mathrm{BiModInt} \circ \mathrm{Oub} \colon \mathrm{BiMod}_{\mathscr{C}^T}(\mathcal{C}, J) \to \mathrm{Alg}(\mathrm{Int}(R_*\mathcal{C}))$$

*is faithful and has a faithful left adjoint.*

*Proof.* By Lemma B.3.11 we know already that Réc $\dashv$ Oub are both faithful, so it remains to account for BiModInt of Lemma B.2.7. Although BiModInt is a left adjoint, it is a reflector. As we observed in Lemma B.3.14, the essential image of Oub is the same as the essential image of AlgExt so that restricted to this BiModInt is in fact an equivalence. Thus the overall composite is a right adjoint as claimed, and both adjoints are faithful. ∎

*Remark* B.3.16. In the most degenerate case of Theorem B.3.15, when $T = \mathrm{id}$, we have in fact constructed an equivalence between algebras for a $\mathscr{C}$-category $\mathbb{C}$ and discrete op-fibrations for the internal category $\mathrm{Int}\,\mathbb{C}$ in $\mathscr{C}$.

To see this we may inspect at first Lemma B.3.1 and the decomposition of $R$ as $\mathscr{C}^T = \mathscr{C} \cong \mathscr{C}/1 = \mathscr{C}/T1$ so that $R$ is the embedding of the full sub-virtual double category on 1 into $\mathbb{H}\text{-}\mathrm{Kl}(\mathbb{S}\mathrm{pan}\,\mathscr{C}, T) = \mathbb{S}\mathrm{pan}\,\mathscr{C}$ and $R_*J = I$ whence $\phi = \mathrm{id}_I$. Thus, by the 2-functoriality of taking bimodules (Lemma B.2.2), the functor $\mathrm{BiMod}_R$ mediates an equivalence with its essential image. But as we argued in Lemma B.3.14, the essential image of $\mathrm{BiMod}_R$ is the same as that of AlgExt. When $T1 = 1$ by Lemmas B.2.7 and B.3.11 we know that AlgExt and BiModInt mediate an equivalence, and so do Oub and Réc. Thus $\mathrm{BiModInt} \circ \mathrm{Alg}_R \colon \mathrm{BiMod}(\mathbb{C}, J) \to \mathrm{Alg}(\mathrm{Int}\,\mathbb{C})$ is the composite of two equivalences.

Considering Theorem 2.2.11, we understand $\mathrm{Alg}(\mathrm{Int}\,\mathbb{C}) = \mathrm{DOpFib}(\mathrm{Int}\,\mathbb{C})$ so that algebras for $\mathrm{Int}\,\mathbb{C}$ are internal discrete op-fibrations. We therefore suggest that our algebra comparison, Theorem B.3.15 above, is a general form of the Grothendieck construction, here witnessed to be an equivalence as expected when $T = \mathrm{id}$. ◁

Our interest in this functor is its application in Section 5, namely, its specialisation to the case of $\mathscr{C} = \widehat{\mathbb{G}}$ and $T$ the free strict $\omega$-category monad. In this case, a $\mathscr{C}^T$-category $\mathbb{C}$ is a category enriched in $\omega$-categories, that is, an $\omega$-category. Moreover, $\mathrm{BiMod}_{\mathscr{C}^T}(\mathbb{C}, J)$ is the usual category of algebras for an $\omega$-category. We therefore record the simplification.

**Lemma B.3.17.** *Let* $\mathbb{S} \colon \omega\mathrm{Cat} \to T\text{-}\mathrm{Cat}$ *be the composite* $\mathrm{Int} \circ R_*$. *Then there is a faithful right adjoint functor* $\mathrm{Alg}_{\omega\mathrm{Cat}}(\mathbb{C}) \to \mathrm{Alg}(\mathbb{S}\mathbb{C})$ *whose left adjoint is also faithful, for every* $\omega$*-category* $\mathbb{C}$. ∎

## B.4. Notes

The author is indebted to Lyne Moser for suggesting a result which is a minor specialisation of the conclusion of Remark B.3.16. This result, as well as the demands of understanding the construction of Section 5.4, led to the present work.

The result of Moser is obtained in the context of that remark when additionally $\mathscr{C}$ is cartesian closed so that algebras for a $\mathscr{C}$-category $\mathbb{C}$ amount to enriched copresheaves $\mathscr{C}\text{-}\mathrm{Cat}(\mathbb{C}, \mathscr{C})$. Moser suggested a direct proof that $\mathscr{C}\text{-}\mathrm{Cat}(\mathbb{C}, \mathscr{C})$ is equivalent to $\mathrm{DOpFib}(\mathrm{Int}\,\mathbb{C})$—an even more striking Grothendieck construction. Our contribution here is to factor the latter part through bimodules and algebras at the level of virtual double categories.

**Improving** $\mathbb{V}$-Cat The present phrasing of Lemma B.1.7 may be altered somewhat so as to suggest that a broader result might be true. Observe that our category $T$-Cat is in fact the vertical category of the virtual double category of modules in $\mathbb{H}$-Kl($\mathbb{S}$pan $\mathscr{C}, T$), that is, $T$-Cat is the vertical category of $\mathbb{K}$Mod$(\mathscr{C}, T) :\equiv$ $\mathbb{M}$od($\mathbb{H}$-Kl($\mathbb{S}$pan $\mathscr{C}, T$)). We already give the definition of bimodule for a pair of virtual double functors, and so one might imagine extending the definition of transformation to allow for arbitrarily many, compatible bimodules in the domain as well as mediating functors of the enriched categories on the extremities. That is, we might construct not just a category of $\mathbb{V}$-categories, but a virtual double category. It is hoped that by doing so we may unify the treatment of Int and BiModInt to a single virtual double functor Int: $\mathbb{V}$-Cat $\to$ $\mathbb{K}$Mod$(\mathscr{C}, T)$. It is perhaps not immediately clear in this context what the role of AlgExt and $\mathrm{Ext}_{\mathcal{C}}$ should be, but certainly a promising avenue for future work awaits.

**On the deficiencies of** $\mathrm{Ext}_{\mathcal{C}}$ In relation to the above, no doubt the curious reader has found our treatment of $\mathrm{Ext}_{\mathcal{C}}$ lacking. We did not address how the functor $\mathrm{Ext}_{\mathcal{C}}$ varies with $\mathcal{C}$ or indeed whether a more general construction is possible in which we do not fix a single $\mathbb{H}$-Kl($\mathbb{S}$pan $\mathscr{C}, T$)-category $\mathcal{C}$. Lacking also is a proof that $\mathrm{Ext}_{\mathcal{C}}$ participates in any form adjunction whatsoever. We invite such a reader to work out these details, with or without us, but to let us know what they find!

**Comparing algebras for externalisation** Our approach here has been rather one-sided. We compared BiMod$(\mathcal{C}, I)$ with Alg(Int $\mathcal{C}$) but neglected to compare the other halves, viz., BiMod$(\Pi \circ \mathrm{Ext}_{\mathcal{C}}\, \mathbb{A}, I)$ and Alg $\mathbb{A}$. This choice should not be taken to mean that we think it less likely that there are interesting results to be had. Rather our course is a simple consequence of expedience: our motivating application does not involve such comparisons. The reader who is interested by such a comparison is once more invited to explore and to share their thoughts.

**On the lack of a general theory for algebras** A final challenge, sidestepped by our at best pragmatic and at worst irresponsible approach, is the lack of a general theory to determine when and how contravariant actions on bimodules may be trivialised so as to recover algebras. At a glance, and with Counter-example B.2.3 in mind, a sufficient treatment might be obtained from exposing the “monoidal” structure of the virtual double category in question—although a framework for such has yet to be developed.

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