The Geodesic Approximation and the L^2-geometry of Vortex Moduli Spaces

Gautam Chaudhuri

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Introduction

A Mechanical Analogy

- Two particles in $\mathbb{R}^2$ joined by a spring of rest length ℓ. A configuration is a point $q \in \mathcal{M} := \mathbb{R}^4$.
- System has Lagrangian,

$$L(q, \dot{q}) := \frac{1}{2}\|\dot{q}\|^2 - \underbrace{\frac{1}{2}\left(\ell - \left\|\left(q^1 - q^3, q^2 - q^4\right)\right\|\right)^2}_{=:V}$$

- Vacuum manifold: $\mathcal{V} := V^{-1}(0) \subset \mathcal{M}$, $\mathcal{V} \cong \mathbb{R}^2 \times S^1$

A Mechanical Analogy

- Suppose $\gamma : [0,1] \to \mathcal{M}$ is s.t. $\sup_t \frac{1}{2}\|\dot{\gamma}(t)\|^2 \ll 1$
- Model γ as $\eta + \varepsilon$ where $\eta : [0,1] \to \mathcal{V}$
- Dynamics of η are free dynamics on $\mathcal{V}$,

$$\{\text{“slow” trajectories in } \mathcal{M}\} \leftrightsquigarrow \{\text{geodesics in } \mathcal{V}\}$$

- How do we apply this to solitons classical field theories?

Plan for talk

- Vortices in Abelian Yang-Mills-Higgs theory
    - Bogomolny equations for vortices
    - Geodesic approximation for vortices
- The Vortex Metric
    - Vortices on a Riemann surface
    - Metric construction
- Geometry of the Moduli Space
- (time permitting) Features of the metric and new results

Vortices

Vortices in Abelian Yang-Mills-Higgs theory

- **Higgs field**: $\phi : \mathbb{R}^{1,2} \to \mathbb{C}$
- **Gauge field**: $a_\mu : \mathbb{R}^{1,2} \to \mathbb{R}$ $(\mu \in \{0, 1, 2\})$
- Lagrangian density,

$$\mathcal{L}(\phi, a_\mu) := -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} \overline{D_\mu \phi} D^\mu \phi - \frac{\lambda}{8}\left(\tau - |\phi|^2\right)^2 \quad (1)$$

  where:

  - $D_\mu := \partial_\mu - \mathrm{i} a_\mu$,
  - $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu$, and
  - $\lambda, \tau \in \mathbb{R}_{\geq 0}$

Vortices in Abelian Yang-Mills-Higgs theory

$$\mathcal{L}(\phi, a_\mu) = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} \overline{D_\mu \phi} D^\mu \phi - \frac{\lambda}{8} \left(\tau - |\phi|^2 \right)^2$$

- $\mathcal{L}$ invariant under U(1)-gauge action,

$$e^{\mathrm{i}\eta} \cdot (\phi, a_\mu) = (e^{\mathrm{i}\eta} \phi, a_\mu + \partial_\mu \eta)$$

- Action functional,

$$S[\phi, a_\mu] := \int_{\mathbb{R}^{1,2}} \mathcal{L}(\phi, a_\mu) \, \mathrm{d}t \, \mathrm{d}x^1 \, \mathrm{d}x^2 \qquad (2)$$

- A **vortex** is a non-trivial minimiser of this functional.

Vortices in Abelian Yang-Mills-Higgs theory

$$\mathcal{L}(\phi, a_\mu) = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi} D^\mu \phi - \frac{\lambda}{8}\left(\tau - |\phi|^2\right)^2$$

- Kinetic and potential terms ($i \in \{1, 2\}$),

$$T = \frac{1}{2}\sum_i e_i e_i + \frac{1}{2}\overline{D_0\phi} D_0\phi$$

$$V = \frac{1}{2}B^2 + \frac{1}{2}\sum_i \overline{D_i\phi} D_i\phi + \frac{\lambda}{2}\left(\frac{1}{2}\left(\tau - |\phi|^2\right)\right)^2$$

  where $e_i = f_{0i}, B = f_{12} = -f_{21}$
- Finite action (at a given time) implies $|\phi|^2 \to \tau$ and $B, D_i\phi \to 0$ as $x \to \infty$

Bogomolny Bound

- Critical coupling: $\lambda = 1$
- Can rewrite:

$$V = \frac{1}{2}\left(B - \frac{1}{2}\left(\tau - |\phi|^2\right)\right)^2 + |(D_1 + \mathbf{i}D_2)\phi|^2$$
$$+ \tau B - \mathbf{i}\,\mathrm{curl}\begin{bmatrix}\overline{\phi}D_2\phi \\ \overline{\phi}D_1\phi\end{bmatrix}$$

- Last terms are entirely topological:
  - $\int_{\mathbb{R}^2} B\,\mathrm{d}x^1\,\mathrm{d}x^2 = 2\pi N$ (not obvious, will see why later)
  - $\int_{\mathbb{R}^2} \mathrm{curl}\begin{bmatrix}\overline{\phi}D_2\phi \\ \overline{\phi}D_1\phi\end{bmatrix}\mathrm{d}x^1\,\mathrm{d}x^2 = 0$

Bogomolny Bound

Theorem (Bogomolny [3], [6, p. 54])

- $\int_{\mathbb{R}^2} V \, \mathrm{d}x^1 \, \mathrm{d}x^2 \geq \pi \tau N$
- *Bound met precisely when,*

$$\bar{\partial}_a \phi := (D_1 + \mathrm{i} D_2)\phi = 0 \quad (3)$$

$$B = \frac{1}{2}(\tau - |\phi|^2) \quad (4)$$

A static field satisfying the vortex field equations is a *minimiser* of the action (not just a stationary point as guaranteed by the E-L equations).

Moduli Space

Theorem

There is a one-to-one correspondence between static vortex solutions (up to gauge) and points in $\mathbb{C}^N/\mathfrak{S}^N$.

Proof (sketch). [6], [7].

- Let $e^{\mathrm{i}\chi} = \phi/|\phi|$
- $\lim_{r\to\infty} |\phi|^2 = \tau \implies \phi \to \sqrt{\tau} e^{i\chi_\infty}$, $\chi_\infty : S^1 \to S^1$
- $\lim_{r\to\infty} D_i\phi = 0 \implies a_\theta \to \partial_\theta \chi_\infty$
    - $\implies \chi_\infty \sim N\theta$
    - $\implies \int_{\mathbb{R}^2} B \, \mathrm{d}x^1 \, \mathrm{d}x^2 = \int_{S^1} \partial_\theta \chi_\infty = 2\pi N$

Moduli Space (cont.)

Proof (sketch).

- For any bdd. simply conn. domain, Ω, $\chi_{\partial\Omega} \sim N_\Omega \theta$
- **Claim**: ϕ is locally gauge equivalent to a degree N polynomial
- $\bar{\partial}$-Poincaré lemma: $\exists u, \bar{\partial} u = \mathbf{i}(a_1 + \mathbf{i}a_2)$
- $\bar{\partial}(e^{-u}\phi) = e^{-u}\bar{\partial}\phi - e^{-u}\mathbf{i}(a_1 + \mathbf{i}a_2)\phi = e^{-u}\bar{\partial}_a\phi = 0$
- $e^{-u} \neq 0$, ϕ is smooth, so $e^{-u}\phi$ is complex analytic (indeed a polynomial)

Moduli Space (cont.)

Proof (sketch).

- $\chi_{\partial\Omega} \sim N_\Omega \theta \implies \deg e^{-u}\phi = N_\Omega$
- (ϕ, a_i) is locally determined by its zero set
- $\chi_\infty \sim N\theta \implies \phi$ only has N zeros globally
- $\{p_1, \ldots, p_N\} \in \mathbb{C}/\mathfrak{S}^N$ determines ϕ up to gauge

$\square$

Moduli space of vortices, $\mathcal{M}_N$ (or just $\mathcal{M}$)

Geodesic Approximation

- Path of static vortices $(\phi(t), a_i(t)) : [0,1] \to (\mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2)$, $a_0 = 0$
- $V(\phi, a_\mu) = \pi\tau N$, $T(\phi, a_\mu) = \frac{1}{2}((\dot{a}_1)^2 + (\dot{a}_2)^2) + \frac{1}{2}\left|\dot{\phi}\right|^2$
- Thus,

$$S[\phi, a_\mu] = -\pi\tau N + \int_{[0,1]} \int_{\mathbb{R}^2} T(\phi, a_\mu) \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}t$$

- $\sup_t \int_{\mathbb{R}^2} T(\phi, a_\mu) \, \mathrm{d}x^1 \, \mathrm{d}x^2 \ll 1$ suggests (ϕ, a_i) should be *close* to a stationary solution

Geodesic Approximation

- Free motion of static vortices with low $\int_{\mathbb{R}^2} T(\phi, a_\mu)\,\mathrm{d}x^1\,\mathrm{d}x^2$
- Not gauge invariant so doesn't define dynamics on $\mathcal{M}$
- Will need to include gauge invariance to construct a metric on $\mathcal{M}$

Riemann Surfaces and the Bradlow Bound

- Consider static vortices on a Riemann surface Σ.
  - Local model is the same as before
  - Additional requirement,

$$\int_{\Sigma} B \,\mathrm{dVol} = \frac{1}{2}\int_{\Sigma}(\tau - |\phi|^2)\,\mathrm{dVol}$$

  - Bradlow bound [4],

$$\varepsilon := \tau \,\mathrm{Vol}\,\Sigma - 4\pi N \geq 0 \tag{5}$$

- $\mathcal{M} \cong \Sigma^N/\mathfrak{S}^N$
- Σ equipped with canonical anti-symmetric tensor ω_{Σ}, Λ is projection onto this tensor

Constructing a Metric

- $[\phi, a_i] \in \mathcal{M}$ a g.e.c. of vortices
- Perturbation $f \oplus \alpha_i \in C^\infty(\Sigma, \mathbb{C}) \oplus C^\infty(\Sigma, \mathbb{R}^2)$
- Needs to solve *linearised field equations*,

$$\begin{aligned} \bar{\partial}_a f - \mathrm{i}(\alpha_1 + \mathrm{i}\alpha_2)\phi &= 0 \\ \Lambda(\underbrace{\partial_i \alpha_j - \partial_j \alpha_i}_{=:\mathrm{d}\alpha}) &= \mathrm{i}\,\langle f, \phi \rangle \end{aligned} \tag{6}$$

  $\langle a, b \rangle = \mathrm{Re}\,\bar{a}b$
- Inner product,

$$(\!(f \oplus \alpha_i, g \oplus \beta_i)\!) := \int_\Sigma \left(\sum_i \langle \alpha_i, \beta_i \rangle_{g^\Sigma} + \langle f, g \rangle \right) \mathrm{dVol} \tag{7}$$

Constructing a Metric

- Infinitesimal gauge transformation: $\mathbf{i}\eta\phi \oplus \partial_i\eta$ for $\eta \in C^\infty(\Sigma)$
- Should project orthogonal to these when defining metric on $\mathcal{M}$
- Projection of $f \oplus \alpha_i$ onto i.g.t.s is given by η solving,

$$\Delta\eta + |\phi|^2\eta = \langle f, \mathbf{i}\phi\rangle - \Lambda(\mathrm{d}\alpha) \tag{8}$$

- The *vortex metric* $g^{\mathcal{M}}$ is given by,

$$(\!(f \oplus \alpha_i - \mathbf{i}\eta\phi \oplus \partial_i\eta, g \oplus \beta_i - \mathbf{i}\nu\phi \oplus \partial_i\nu)\!) \tag{9}$$

Geometry of the Moduli Space

Metric Geometry

- $(\mathcal{M}, g^{\mathcal{M}})$ is geodesically complete
- $(\mathcal{M}, g^{\mathcal{M}})$ is known to be Kähler [1], [10]:
  - Σ a compact Riemann surface $\implies$ $\mathcal{M}$ compact
  - Can calculate $\operatorname{Vol} \mathcal{M}$
- Closed form expression for the metric is not known, need to localise/approximate
- Localisation formula due to Samols [10] yields expression in a nbhd. of $\{p_1, \ldots, p_N\} \in \mathcal{M}$

The Dissolving Limit

- Suppose genus of Σ is $g \geq 0$,
- $\varepsilon = \tau \operatorname{Vol} \Sigma - 4\pi N \geq 0$:
    - $\varepsilon > 0$, $\dim \mathcal{M} = N$,
    - $\varepsilon = 0$, $\dim \mathcal{M} = g$
- Increase in symmetry since $\varepsilon = 0 \implies \phi \equiv 0$
- Can we approximate $g^{\mathcal{M}}$ in the $\varepsilon \to 0$ limit?

Fibre Bundle Structure of the Moduli Space

- $\mathsf{Jac}\,\Sigma \cong \mathsf{U}(1)^g$ is a natural Kähler manifold attached to Σ
- Smooth map $\pi : \mathcal{M} \to \mathsf{Jac}\,\Sigma$:
  - $N + 2 - 2g \leq 0$, immersion, approximate in $\varepsilon \to 0$ by metric on $\mathsf{Jac}\,\Sigma$ [8], [9]
  - $N + 2 - 2g > 0$, submersion, fibre over a point is $\cong \mathbb{CP}^{N-g}$
- $g = 0$: $\mathsf{Jac}\,\Sigma \cong \{\mathrm{pt}\}$, $\mathcal{M} \cong \mathbb{CP}^N$ *Conjecture*: $g^{\mathcal{M}}$ well approximated by Fubini-Study metric on $\mathbb{CP}^N$ as $\varepsilon \to 0$ [2]

New Results

C0 Convergence (on Fibres)

Theorem (García Lara and Speight [5])

If $\Sigma \cong S^2$, then $\left|\frac{1}{\varepsilon} g^{\mathcal{M}} - g^{\mathrm{FS}}\right|$ is bounded above by $C\varepsilon$ in C^0

Corollary

On every fibre $\pi^{-1}(p)$, $\left|\frac{1}{\varepsilon} g^{\mathcal{M}}\Big|_{\pi^{-1}(p)} - g^{\mathrm{FS}}\right|$ is bounded above by $C\varepsilon$ in C^0

C1 Convergence (on Fibres)

Theorem (to appear)

Σ s.t. $N + 2 - 2g \geq 0$, then on every fibre
$\left| \frac{1}{\varepsilon} g^{\mathcal{M}} \big|_{\pi^{-1}(0)} - g^{\mathrm{FS}} \right|$ is bounded above by $C\varepsilon$ in C^1

Corollary

Geodesics on fibres are well approximated by geodesics of the Fubini-Study metric.

The Large Degree Limit for Homogeneous Surfaces

- Ongoing work: how does the constant C depend on N?
- Studying proof suggests $C \sim e^N$
- Numerical work on Σ the unit sphere, suggests C grows unbounded

References

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